

Problems and Problem Solving

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The purpose of this document is to provide guidance and support to teachers from the Early Childhood level through to Grade 6 in the appropriate and effective way to prepare children and pupils to become numerate. The National Comprehensive Numeracy Programme with this document, therefore, will provide the context and approach to be employed in mathematics instruction throughout these levels; so as to set the right foundation upon which to develop the skills and competencies needed for students to be successful in mathematics, and to display adequate and appropriate numeracy skills in their everyday life.

The document establishes the approach to be employed which is founded on three fundamental principles:

- 1. Conceptual Understanding
- 2. Computational Fluency
- 3. Problem Solving

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INTRODUCTION

Overview

According to the National Council of Teachers of Mathematics (NCTM), problem solving is one of the important processes in which students should engage as they interact with the content strands in the curriculum. The Ministry's Revised Primary Curriculum for Mathematics has many objectives that are explicitly related to problem solving. For example, from as early as Grade 1, students are expected to "solve simple problems including the use of money" and at Grade 6, they are expected to "represent and solve problems using geometric models". Other objectives, while not as explicit, allow teachers to design tasks that encourage students to reason, hypothesize, strategize and, in short, to problem solve.

This handbook has been developed by the National Mathematics Team of the Ministry of Education because much anecdotal evidence suggests that there has been a dearth of problem-solving activities in mathematics classes and that there are insufficient non-routine mathematics problems available to teachers. In addition, problems identified in textbooks are usually 'dressed-up **algorithms**' and by themselves are inadequate to develop deep critical thinking in students. It is against this background that the Team has developed this handbook, which seeks to:

- a) provide a clear understanding of what is meant by 'problems' and 'problem solving' in mathematics;
- b) build the capacity of practitioners to develop non-routine mathematics problems as a part of their teaching and assessment of mathematics;
- c) empower teachers and through them, students to solve problems and to develop a bank of heuristics on which they draw habitually when confronted with mathematics problems;
- d) assist teachers in planning lessons that embrace a problem-solving approach to teaching; and
- e) provide a bank of age- and grade-appropriate problem-solving activities and tasks from which teachers can draw for use in the mathematics class.

The philosophy behind the development of this handbook, therefore, goes beyond merely providing teachers with problems that they can use in their classes; rather, the handbook was created to address many of the other areas and issues related to the use of problem solving. In the handbook, you will find:

- a) a sample problem, solved using three different methods with helpful notes as to the thought processes involved in exploring each method;
- b) fully worked and explained problems showing how different strategies can be used to solve problems;
- c) examples and non-examples of genuine non-routine mathematical problems;
- d) a sample rubric, applied to three examples of a solved problem, that shows how performance on problem-solving tasks can be assessed and interpreted by teachers;
- e) suggestions regarding the grade level and how students should be organised when asked to solve each problem in the problem bank;
- f) that each problem is linked to appropriate strands and concepts in the curriculum; and
- g) a comprehensive glossary, at the end of the handbook, of important words and terms that are highlighted in the text.

To Teachers

At times, out of utter frustration, you may be tempted to say that your 'students can't think'. Indeed, there is often an absence of thought in many mathematics classrooms; however, it is quite likely that your students will think if they are given the opportunity to or if they are exposed to a thought-based classroom, over time. As you introduce some of these problems to your students, you may find them becoming frustrated as they grapple with tasks that are novel and for which they do not have worked examples, or for which they are unable to find solutions. Your response at this stage is crucial. **DO NOT BE ALARMED IF THEY ARE FRUSTRATED**. In time, as they come to appreciate that there is a difference between 'not knowing' and 'not having found out as yet', the frustration is likely to be replaced by enthusiasm. Do not give them answers or show them worked examples; you may, however, provide useful prompts to point them in a particular direction or you may ask open questions to encourage them to think about the problems in different ways. Remember that your role is to help students to develop the ability to solve problems by themselves.

Most importantly, **DO NOT DECIDE TO KEEP A PROBLEM FROM YOUR CLASS BECAUSE YOU THINK THEY CANNOT DO IT.** Do not create a climate in your classroom in which right answers are more important than strategizing, or in which students think they must be able to solve all problems fully. When alternative solutions are presented by students, admit that you had not

seen it that way; explore why this solution is also true (or not) and, by doing this, empower students to share answers that are different from others.

A few teachers might also experience challenges in solving problems and are themselves likely to become easily frustrated. In these instances it is important to change perspective and persist in attempts to solve the problem. Exploring and building on what must be true about the problem is also a good approach.

We hope that you will find this handbook a source of support for yourself and your students and that, through it, the aim of improved performance will be realised.

Seymour Hamilton, National Mathematics Coordinator

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SECTION 1 CLARIFYING PROBLEMS AND PROBLEM SOLVING

Problems and Problem Solving

A cursory perusal of the mathematics content outlined in the Revised Primary Curriculum (RPC) reveals that the terms '**problems'** and '**problem solving**' are introduced as early as Grade 1 and used with increasing frequency thereafter. This sharp focus on problem solving in the RPC reflects the fact that, in recent times, greater significance has been placed on curricula that produce students who are able to solve problems in the real world. These terms – 'problem' and 'problem solving' – however, require some clarification. Below, we offer four definitions from various sources.

Definition 1

NCTM (2000, 24) sees a problem as:

a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings.

Definition 2

Erickson (1999, 516) describes problems as:

situations in which no readily known or accessible procedure or algorithm determines the method of solution. The problem may come from a real-world application or a puzzling dilemma. The problem or task should be an activity that focuses students' attention on a particular mathematics concept, generalization, process, or way of thinking that matches the goals of school mathematics.

Definition 3

Hiebert, et al. (1997, 75) explains a problem as:

a task or activity for which the students have no prescribed or memorized rules or methods, nor is there a perception by students that there is a specific 'correct' solution method.

Definition 4

Foong, P.Y. (2001) put forward a definition of a problem situation as: one where thinking takes place when a person is confronted with a [task] that has no immediate solution and that the problem-solver accepts the challenge to tackle it. Bearing in mind the definitions above, note the following points.

Bearing in mind the definitions above, note the following points.

- a) Problems allow students to do more than simply recall stored facts and/or procedures in order to obtain answers; rather problem-solvers use critical thinking to devise solution methods by drawing on **and** applying their knowledge in order to produce results.
- b) Problems allow students to make sense of mathematical situations for which they have no well-defined, rehearsed, standard routines or procedures.
- c) Problems are tasks that allow students to develop new insights into, or a sophisticated understanding of, mathematical concepts or procedures arising from or associated with these concepts.

Types of Problems

Broadly speaking, there are two types of problems.

Routine problems

According to Gilfeather and Regato (1999), routine problems are tasks which use up to three steps to facilitate the use of known, or prescribed, procedures in obtaining solutions. Routine problems are generally described as being well-structured - the tasks are clearly formulated and, typically, use traditional 'worded problems' to determine students' ability to use standard algorithms. Algorithms can be seen as:

- rules for calculating;
- computational procedures for deriving a solution to a given problem; and
- logical step-by-step procedures for solving a mathematical problem. Sometimes they involves the repetition of the same steps.

Algorithms

Algorithms are suitable for solving routine problems such as:

- $(2+6) \times 3$ a)
- b) What is the area of a rectangle with a length of 12 cm and a width of 9 cm?
- c) $134 \div 14$
- Determine the value of x if 2x + 3 = 9d)
- $\frac{1}{4} + \frac{2}{3}$ e)

For each of these examples, it is likely that students would have been exposed to known algorithms that they could use to obtain solutions. Therefore, in assessing the responses of a class of students who attempted these questions, one expects that most of them would have employed the following algorithms:

- BOMDAS or BODMAS
- 'Length × width'
- 'Long' division
- Grouping like terms; 'changing the sign' when variables or numbers 'go across the equal sign'
- Finding the LCM

Non-routine problems

These are problems for which no known algorithm exists, or is thought to exist, that can provide the solution to the problem. These problems are solved through the use of **heuristics** and not algorithms. According to Gilfeather and Regato (1999, 20):

Heuristics are procedures or strategies that do not guarantee a solution to a problem but provide a more highly probable method for discovering [its] solution. Building a model and drawing a picture of the problem are two basic problem-solving heuristics.

An example of a non-routine problem is given in the coin problem below:

If you had six coins in your pocket of the following denominations: 10¢, 25¢, \$1, \$5, \$10, \$20 How many monetary values could you make?



How is this problem different from the routine problems that we looked at before?

- a) No clear or prescribed method or approach to arriving at the solution is implied in the question. It is likely that students have never been taught a method of determining the total number of monetary values that can be made up from a set of coins. The problem is novel and no approach is specified or readily identifiable; hence, the problem solver will have to devise his or her own method of obtaining an answer.
- **b)** The question engages students in critical thinking. They think about and attempt to develop a strategy to obtain a solution; they draw conclusions, inferences, conjectures, and develop and test hypotheses.
- c) Students' solutions may vary. While all the answers are expected to be the same, students' strategies may differ. Some solutions will be more sophisticated than others – some students may try many approaches and not arrive at an answer, while others may find methods that work to obtain an answer and to gain insight into the problem.
- d) The task encourages communication and collaboration. Given that no clear procedure exists for solving the problem, it is likely that students will have to collaborate with each other and engage themselves in mathematical discussion as they seek solutions and attempt to communicate these results for others to understand.
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Features of Non-routine Problems

Non-routine problems are identified by the features of openness, critical thinking and novelty. We look at each of these features below.

a) Open – When we say a problem is 'open', we mean that either the approach to be taken to solve the problem or the answer to be obtained at the end of the solution process can differ from one student to the next. Look at the two questions below and observe how different one is from the other in terms of its openness.



- **b) Critical thinking** Non-routine problems put students in different, and often ambiguous, situations that require them to invent well-reasoned approaches to arrive at solutions.
- c) Novel non-routine problems are different from routine problems in two main ways:
 - i. Non-routine problems are not always linked to content in the same way that regular routine problems are. A child may solve a non-routine problem without realizing that he or she is applying one of the standard topics in the curriculum. The coin problem previously mentioned is one such example. These problems allow students to develop their critical thinking skills, which allow them to access content outlined in the curriculum more easily.
 - ii. Even when non-routine problems are linked to the content outlined in the curriculum, they take a fresh and original approach to the exploration of this content. So, for example, instead of asking students to find the perimeter of a rectangle, a problem may be posed to ask them to explore the different perimeters possible, given that the rectangle has a fixed area of 24 cm².

Given the reasons outlined above, non-routine problems have been identified as having tremendous potential to impact students' ability to adapt to real-world situations. *Santos-Trigo and Camacho-Machin* (2009) make the important point that real-world problems do not come as nicely packaged and highly abstracted as those that students frequently encounter in routine classroom problems; rather, they are characterized by ambiguity, missing or unknown

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information and are usually solved after much analysis and critical thinking. A curriculum of routine problems, therefore, does not produce students who are likely to excel in situations where strategizing and thinking will be needed. It is for this reason that non-routine problems are important and why this handbook seeks to encourage their use in the classroom.

Types of Non-routine Problems

Non-routine word problems

- These are problems that require students to read, interpret and resolve problem situations using information given explicitly in the question as well as that which is only implied.
- These problems are resolved by identifying key pieces of information in a problem and making inferences based on them. Most non-routine word problems are based on real-life situations (even though some amount of abstraction may be done).
- A sample of a non-routine word problem is shown in the census taker problem below (adapted from London, 1993).

A census taker comes to the house of a mathematician and asks how many children he has and what their ages are. The mathematician replies that he has three children and the product of their ages is 72. The census taker replies that he has not been given enough information to determine their ages. The mathematician adds that the sum of their ages is the same as his gate number. The census taker leaves to check the gate number but returns shortly to say that he still does not have enough information. The mathematician thinks and says the oldest one likes chocolate ice cream. The census taker replies that he now has enough information and leaves. What are the ages of the three children?

Having read the problem, consider the following questions.

- a) Of what relevance is it that *after* the census taker checked the gate number he realized that he did not have enough information?
- b) What is the important mathematical fact in the observation that the oldest child likes chocolate ice cream? Is the more important clue the fact that the child likes chocolate ice cream or that there is an 'oldest' child?

In looking at the census taker problem and on reflecting on our previous discussion about the creation of original problems, it is clear that the problem was created by exploring a real-world

situation in which a census taker seeks to obtain information from a 'difficult' mathematician. The problem situation changes the nature of the interaction between the mathematician and the census taker without making it impractical or absurd. This changing of the regular interaction is called abstraction and is usually necessary when real-world problems are being used in the classroom.

Process problems

These problems are usually written in worded form but are significantly different from common worded problems that are typically solved through the application of known facts or algorithmic routines. Process problems require the application of problem-solving strategies other than computational processes and may lead the problem solver into exploring many ideas by developing and testing hypotheses.

Usually, obtaining the solution to a process problem will require some use of generalization, pattern spotting or **predicting**. An example of a process problem is shown in the popular handshake problem below.



The sixth grade girls' netball team at James Town Primary won the Regional Championship. All 12 members of the team (players along with reserves) congratulated each other with a handshake. How many handshakes were exchanged?

While computation will be involved in 'counting up' the number of handshakes exchanged, problem-solving skills such as generating and identifying number patterns and sophisticated counting strategies (in order to avoid handshakes being unaccounted for or being counted more than once) will also be needed.

In creating process problems, therefore, a useful source is to use patterns that your students can identify, describe and apply. Indeed, as you develop process problems, you will find that you are most likely to develop most of your problems by exploring patterns with your students.

Open problems

Open problems are those in which the tasks are 'ill-structured' or ambiguous as they have missing information and, hence, force the students to make assumptions and decisions about including additional facts and figures.

What usually makes these tasks problematic is the thought process that students go through in attempting to determine the additional information needed to solve the problem and how they

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should go about gathering the needed information – either through making assumptions and/or estimates, carrying out research or by performing calculations.

Open problems usually produce different results, since each student may make different assumptions and estimates or may choose different facts and figures to include in the task. It is important, however, that students are able and allowed to justify their assumptions, estimates and choices in light of the answers that they received.

Open problems are usually derived from two main sources by:

- i. transforming routine tasks into non-routine problems; and
- ii. using real-world situations.

Transforming routine tasks

Enrico Fermi, a mathematician, developed and successfully used open problems with his students. These problems demonstrate how routine tasks can be transformed into open non-routine problems. The table below shows some routine problems in the first column and then some of Fermi's open variations on these in the second column.

Routine problems	Open problems (adopted from Peter-Koop, 2005)				
The polar bear p	roblem				
The average child in a school weighs 35 kg; the polar bear at the zoo weighs 665 kg. How many chil- dren will it take to weigh as much as the polar bear?	How many children are, together, as heavy as a polar bear?				
The water problem					
Each day you use 13 litres of water. How many litres of water do you use in a week?	How much water do you use in 1 week?				
The traffic pro	blem				
How many cars of length 20 m are in a traffic jam that stretches for 3 km, if a space of 6 m is between each car?	There is a 3 km traffic pile up on the high- way. How many vehicles are caught in this traffic jam?				

Fermi's wording forces students to make many assumptions or estimates if they are to attempt to solve the problems. For example, with the traffic problem, students must seek answers to the

following questions.

- Are all vehicles of the same length?
- What is the average length of a vehicle?
- Typically, in a line of traffic, will most vehicles be 'large', 'small' or 'medium size'?
- Approximately how much space would there be between vehicles?

If you look closely at each of Fermi's problems, it is clear that they resemble (but are significantly different from) routine problems. Indeed, it is not hard to imagine that these are simply routine problems that have been modified in one way or another.

Real-world situations

By reading through Fermi's problems, you will also realize that each problem asks students to reflect on and mathematize their real world experiences. So, by placing students in what ought to be familiar or accessible context, you will encourage them to start thinking about and operating with numbers; they will also start to consider whether answers are 'reasonable' by reflecting on their experiences and observations or by carrying out experiments.

Puzzles

A mathematical puzzle is an activity in which, typically, numbers and/or objects are used to complete a task by organizing or using them in a particular way. Puzzles are created by considering **cognitive roots** – core and important ideas that can be extended to generate problems – and by

placing the problem solver in complicated situations where complex relationships must be understood and modeled. Most mathematical puzzles involve one or more of the following.



i. Manipulation of objects to create patterns or designs. In the Rubik Cube puzzle, for example, persons try to rearrange multicoloured blocks that are used to make a cube so that each face of the cube has a separate solid colour. The cube is made up of 54 blocks with a total of six different colours – each face can be

'twisted' so that colours can be aligned. Another example of a puzzle which involves manipulation is the Testa Puzzle on the following page.

Arrangement of numbers according to set rules. Sudoku, for example, requires that players insert 9 ones, 9 twos, 9 threes, ... 9 nines in a large 9 × 9 grid (made up of three 3 × 3 squares) so that no row or column has the same number. Additionally, each 3 × 3 square must have each of the numbers 1 – 9. Another example of a number-based puzzle, The Magic Triangle , is shown on the following page.





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iii. Use of geometric principles and spatial awareness. Some puzzles are built on the problemsolver's ability to apply his or her knowledge of lines, angles and shapes to problem situations; in solving these puzzles, the problem solver may have to draw on his or her spatial awareness and ability to appreciate such principles that re-orientation usually does not change the essential characteristics of a shape. Matchstick problems fall in this category of puzzles that can be solved using geometric principles and spatial awareness. Below, for example, is a popular matchstick puzzle – the 'wine glass' problem.



iv. Logic: Some puzzles are built on logic and the use of deductive reasoning. These puzzles usually give the problem solver all the information that is needed in order to obtain a solution; however, the information must be organized for interpretation. Additionally, these questions usually have many layers: the solution for one part is taken as the starting point for another part. The logic puzzle highlighted below shows these features.

Three cups of different colours (red, blue and green) are on a table; each cup has an object in it – either a coin, a bean or a shell. You are told the following:

To the left of the red cup is the green cup. To the left of the bean is the coin To the right of the shell is the blue cup To the right of the blue cup is the bean

Identify which object is in each cup.

Games

Many problems are derived from game situations. Games that create problems are those in which a winning strategy is to be established by the players so that, regardless of what an opponent does, a particular series of plays will ensure that one person will always win. A winning strategy usually consists of:

- a) a rule as to who plays first; and
- b) a sequence of moves that guarantees a victory for one player.

The derivation of the winning strategy, however, is the search for a solution and will require the players to employ many problem-solving heuristics. The Spiral Game, on the following page, is an example of a mathematics game that is problematic.

Instructions:

- a. This is a game for two players. Each player will take turns in moving a counter around the board from start to end.
- b. A counter is placed on number 1 only one counter is to be used (do <u>not</u> use one counter for <u>each</u> player). Each player will take turns moving the same counter around the board.
- c. The player who chooses to go first will move the counter between 1 and 6 dots
- d. Player 2 now moves the counter from where player 1 leaves it; he also has to move it between 1 and 6 dots.
- e. Each player will continue taking turns moving the counter between 1 and 6 dots along the spiral; turning back is not allowed.
- f. The winner is the player who moves the counter on dot number 25. Try to find a winning strategy.
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Problem solving

Students engage in problem solving when they seek to find solutions to tasks using methods and strategies that they contrive or invent (Salleh and Zakaria, 2009). In other words, problem-solving involves the creation of strategies in order to get to a stated end. Problem solving is an active experience involving cognitive and metacognitive strategies – not only do problem solvers think about the task, they also analyse their thought process by asking questions such as:

- Why is this true?
- Does it always work?

Through problem solving, students can develop new knowledge, solve any problem that occurs, apply and use various strategies, and also reflect on and monitor the problem-solving process.

A detailed example of a problem

The Problem

Suppose that	at you had siz	coins of the	e following de	nominations i	n your poc	ket:
10¢	25¢	\$1	\$5	\$10	\$20	A
How many o	different mor	netary value	s could they n	nake?		

A few observations about the problem

- a) The question will involve low level addition. Some students will add various denominations of coins to get their monetary values. Others will simply identify the various combinations without performing the additions. The question is more than a routine 'addition problem', however, as students will have to:
 - i. devise a system of identifying and counting the sums to ensure that they have not left out any of the possible monetary values; and
 - ii. predict what happens when the number of coins becomes so large that actually forming combinations and counting them become impossible.
- b) The question makes use of realities and experiences that students are likely to have had and can relate to. Students are likely to be familiar with the coins in the Jamaican system and can actually use these coins to physically **model** their solutions.
- c) The use of six coins is deliberate. It is difficult for children to identify the different number of monetary values when coins of many different denominations are being used. More than likely, children will have to devise a system of predicting the number of monetary values once more than four coins are being used.
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Model solutions

Essentially, the approach taken is to simplify the problem by exploring cases with fewer coins. The hope is that we will spot a pattern that allows us to predict the results when six coins are being used.

In exploring simpler cases, you will find that counting strategies are very important – students are going to have to devise strategies to ensure that no monetary value is repeated and that no case is omitted.

Below, we show three different counting strategies to ensure that this is done.

Solution 1

- With one coin (only the 10¢), only one monetary value is possible
- With two coins (the 10¢ and the 25¢), the possible monetary values are

10¢ 25¢ 10¢ + 25¢

- With three coins (the 10¢, 25¢ and \$1), the possible monetary values are:
 - o **10¢**
 - o **25¢**
 - o **\$1**
 - **10¢ + 25¢**
 - **10¢ + \$1**
 - **25¢+\$1**
 - **10¢ + 25¢+\$1**
- With four coins, great care has to be taken in listing the coins particularly when coins are to be combined. The system of listing that we will use is shown below:
- Individual coins

10¢	25¢	\$1	\$5
-----	-----	-----	-----

Combinations of two coins

To determine all the possible sums, we will, in each case, hold one coin constant and vary the coin with which it is combined.

• First we will keep the 10¢ constant and combine it with different coins.

• We will then keep the 25¢ coin constant and combine it with the other coins, keeping in mind that it has already been combined with the 10¢.

```
25¢+$1 25¢+$5
```

• Next, we will keep the \$1 coin constant and vary the other coin, but at this point we have already combined the \$1 with the 10¢ and the 25¢, leaving only the \$5.

\$1 + \$5

• Once we get to the \$5 coin, we will find that we have already combined it with the 10¢, 25¢ and \$1 coins, thereby exhausting all possible combinations.

So there are six possible monetary values when two coins are combined.

Combinations of three coins

To determine all the combinations of three coins we will use the combinations of two coins we had previously found as a base and build on it using a method similar to the one used in that section. This time around, however, rather than keeping one coin constant we will keep one pair of coins constant and vary the third coin with which they are combined.

• The first pair we will hold constant is 10¢ + 25¢. This will be combined with each of the two remaining coins.

10¢ + *25¢*+*\$1 10¢* + *25¢*+*\$5*

• We will then move to 10¢ + \$1. Here we will need to exercise some caution, keeping in mind that we would have already seen the combination of 10¢ + 25¢.

10¢ + \$1+ \$5

- Next, we will look at 10¢ + \$5. By now we would have seen that there will be no new combinations as these have already been explored in the previous cases.
- Similarly, we will examine 25¢+ \$1. Once we have done this we should realize that we have exhausted all possible combinations.

25¢+\$1+\$5

So here we have four possible monetary values when three coins are combined.

Combinations of four coins

Immediately we can appreciate that there is only one set of four different coins, and need not reduce the combinations that we had for the previous cases.

\$1 + \$5 + \$10 + \$20

In all, we have a total of 15 possible monetary values when we have four coins.

We can, therefore, put these results in a table as shown below.



The table above shows that the number of monetary values increases in a precise way when a coin of a new denomination is added: the number of **new** monetary values is doubled. It can also be used to show that when:

- there are 5 coins, there are 31 monetary values (15 + 16); and
- there are 6 coins, there are 63 monetary values (31 + 32)

Solution 2

In this method the solution is obtained by creating a pictorial representation of the possible combinations – starting with combinations using two coins.

• With two coins, only three solutions are possible:



• With three coins, the following solutions are possible:

• With four coins, there are 15 solutions. Given space considerations, however, we will only list a few here.

• There are four single coins:

• There are six combinations with two coins (two of them are shown below):

• There are four combinations with three coins (one is shown below)

• There is one combination involving all four coins.

In total, therefore, there are 15 different monetary values when four coins are used. From here, a similar approach as the one taken above in solution 1 can be used to show that there are 63 different monetary values when six coins are being used.

Solution 3

Note that values are made up either by considering a coin by itself or by combining it with others. For example with one coin (\$0.10) only one value of \$0.10 is possible. If two coins (\$0.10 and \$0.25) are considered, however, then in addition to the one value now possible, the following two values are also possible:

- \$0.25
- \$0.35 (the sum of the two coins)

The table below shows how monetary values can be made up by starting with one coin and adding other coins

Number of coins	New values after each new coin	Number of new values	Total number of new values
1 (<i>\$0.10</i>)	\$0.1	1	1
2 (\$0.10, \$0.25)	\$0.25 \$0.35	2	3
3 (\$0.10, \$0.25, \$1)	\$1 \$1.10, \$1.25 \$1.35	4	7
4 (\$0.10, \$0.25, \$1, \$5)	\$5 \$5.10 \$5.25 \$5.35 \$6 \$6.10 \$6.25 \$6.35	8	15

A few things to note about the table:

- 1. New values are determined by adding new coins to values obtained previously. As an example, when three coins are being considered, the new coin being used is \$1. This coin is added to the three values obtained before.
- 2. The total number of values is determined by adding new values to previous values.
- 3. It may not be immediately clear, but in each case when a new coin is added the following happens:
 - a. The total number of values **increases by 1** because of that coin
 - b. The total number of values increases as well **by as many values as existed before**. This is so as the new coin is added to each of the values so far to give as many new values.
- 4. Note that number of new values is the double of the previous number. Alternatively, number of new values is the same as the sequence of squared numbers.

These facts can be used to predict the number of values for six coins as follows:

Number of coins	1	2	3	4	5	6
Number of new values	1	2	4	8	16	32
Total number of values	1	3	7	15	31	63

Rationale for problem solving

Having looked at the problem above in such detail, let us now turn our attention to looking at some of the reasons why problem solving should be such an integral part of the students' class-room experience. The rationale for the use of problem solving in the teaching of mathematics is grounded in the following observations.

- a) Students must solve problems in everyday life and, therefore, mathematics should be taught with this in mind. Real-life problems, however, do not always come 'nicely packaged' as classroom problems are. In real life, the starting point is not always clearly identified, the solution paths are not always immediately obvious and the answer is often ambiguous and subjective. Typical classroom activities, for example, where 'all round shapes are circles' or where one division usually produces answers with no more than two decimal places do not adequately prepare students for open, real-world problems.
- b) Problem solving develops students' ability to think critically. By solving problems, students become better at reasoning, making logical and likely suppositions, communicating and representing their ideas for others to understand, and advancing **proofs** to support their conclusions.
- c) Problem solving develops **volitional** strategies in students. This refers to students' ability to act independently of the teacher or other sources of guidance (such as the textbooks or previously worked examples of similar problems) and create solution paths when faced with novel problems.
- d) Effective, well-managed problem solving classes build confidence in students. When students solve novel problems and develop the tools needed to solve other problems, they are empowered to take on other mathematics tasks.

SECTION 2

PROBLEM CREATION

Creating Original Problems

While a number of mathematics problems are available to teachers via the internet and textbooks, you should be able to develop your own mathematics problems. Following are some factors that you should take into consideration in developing problems.

- **THE GRADE LEVEL OF THE STUDENTS**. Try to pose problems that reflect the grade level of students. Be deliberate and careful in your choice of numbers, the information you share with students and how you ask the questions.
- **THE CULTURAL ENVIRONMENT OF THE STUDENTS.** Try to pose problems that reflect the situations they regularly encounter or the experiences that they are most likely to have had. In a farming community, for example, some problems may be related to farming. Remember though, that with enough background information and depending on how the problem is presented, students may be able to access problems outside of their experience. Consider the following example.

"What number sequence would reflect the number of players for which no byes would be necessary in a tennis tournament?"

- o One quickly realizes that the problem provides insufficient information on what 'byes' are and what a tournament is. The situation is made worse by the fact that tennis is not likely to be a popular sport among many Jamaican children.
- o For an average child, therefore, it is possible that little interest and much confusion will arise upon reading this problem. What if the problem had been presented as follows:

There are six groups in a class at Hampton Primary School. They are having a quiz competition in which the groups are expected to compete. Winners will advance to another round and losers 'drop out'. This continues until only one team remains.

The teacher has prepared the following schedule:

Round 1 Match 1: Group A vs. Group B Match 2: Group C vs. Group D Match 3: Group E vs. Group F

In Round 2, however, the principal realizes that one group will not have a playing partner, since only three teams would have advanced from Round 1 and so he decides that the 'best loser' from Round 1, will have to be allowed to play as well – this team is said to have gotten a bye (a free pass into round 2). The schedule for Round 2 is as follows:

Round 2 Match 4: Winner of Match 1 vs. Winner of Match 2 Match 5: Winner of Match 3 vs. best loser from Round 1.

Of course in **Round 3**, only two teams remain and the winner will be determined from those.

a) For which of the following number of players will byes be needed in a round?

- i. 10 players
- ii. 8 players
- iii. 18 players

b) Can you think of some numbers for which there will be no byes in any round?

- c) In general, what numbers will always result in no byes being necessary?
- You can see from this example that by changing the context within which problems are posed, they are made more accessible.
- PROBLEMS SHOULD BE GEARED TOWARD ENCOURAGING CHILDREN TO THINK. These
 problems should be tailored to give students the opportunity to explore a variety of solutions
 and think critically. This can be facilitated by considering the following points.
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- Explore a variety of situations. For example, consider the following two questions:
- 1. Calculate the value of 11×12
- Michelle does not know the answer to 11 × 12. However, she knows that 6 × 12 is
 72; explain to her how she can use this information to determine 11 × 12.

Question 2 allows the student to engage in more thinking and exploration than question 1 does.

- Think in novel ways. For example, by the time students reach Grade 6, they would have encountered two-digit multiplication and squaring of numbers such as 95. In solving problems of this nature-students normally apply the long multiplication algorithm. If, however, they were asked to explore what happens when a number that ends in 5 is squared such as 5², 15² and 25² and then asked to predict the answer for 95², they would, likely, attempt to identify a simpler method for solving these problems.
- PROBLEMS MUST GET STUDENTS EXCITED ABOUT LEARNING MATHEMATICS. Exciting
 problems are those that stimulate students' interest and challenge them to find a solution.
 In these problems items from students' daily experiences are used to generate curiosity and
 excitement.
- CHILDREN'S LITERATURE AND OTHER SUBJECT AREAS SUCH AS SOCIAL STUDIES, LANGUAGE ARTS AND SCIENCE CAN BE USEFUL SOURCES FOR DEVELOPING PROBLEMS. The problem below, for example, shows how language can be incorporated into a mathematics problem.

If you write the three-letter word COD in a grid as shown below, there are two ways of spelling it.

In the following grid, the 4-letter word TAKE can be written in 3 different ways. Determine if this is indeed so.

т	А	К
А	К	E

What about DAILY and SALMON?

D	А	Ι	S	Α	L	М
А	Ι	L	А	L	М	0
Ι	L	Y	L	М	0	Ν

Predict the number of ways in which MATHEMATICS could be written if it were presented in a similar grid.

 PROBLEMS MAY BE PRESENTED SO THAT THEY PROVIDE SCAFFOLDING OR SOME GUIDANCE TO STUDENTS. Providing scaffolding means that we supply students with information about the problem and give a framework for deriving further solutions. This is demonstrated in the way the conversation problem is presented.

A conversation relationship occurs when two persons in a group share information. For example, if there are 2 persons (A and B) in the group, then there is only one possible conversation relationship – A and B will converse with each other. However, if there are 3 persons (A, B and C) in the group, then the following 3 conversation relationships are possible:

- o A with B
- A with C
- **B with C**

This is shown right:

Now, if there are 4 persons in the group, there are 6 possible conversation relationships. Complete the diagram below to verify that this is so.

Now, determine the number of conversation relationships possible if there were 30 persons in a group.

As we can see from the example, the information provided gives the students "a way into the problem". Let us consider the following.

- The first few situations were solved for the students they now know how many conversation relationships are possible when two, three and four persons are in the relationship. They can start looking for patterns at this point and use this pattern to predict the number of conversation relationships possible among five persons.
- Students were shown how to represent the solution for three persons. They were also given an insight into how to proceed for determining the number of possible relationships.
- The answer for a four-person relationship was provided so as to present students with the opportunity to practise the use of the method of their choice. Students are, therefore, given the opportunity to check the accuracy of their methods by ensuring that they, indeed, end up with six possible relationships.

It is important to decide on how much information you will share with the students, as well as how the questions will be structured when designing a problem. For example, in exploring the conversation problem, we expect students to obtain the following results:

Students can fill in the unknown cells by realizing that the value in each cell is obtained by adding consecutive numbers to the number on its left. Using this idea, the following table can be obtained.

No. of Persons	1	2	3	4	5	6	7	8	9	10
No. of Conversations	0	1	3	6	10	15	21	28	36	45

The question asked students to say how many conversation relationships are possible when **30 persons** are present. This could be tedious if they had to continue adding consecutive numbers to count up to 30 persons. We used the number 30 as we wanted to demonstrate to students that this method of 'counting up' to a particular number is time consuming. Here we hope they will realize that they must now think of relationships between the number of persons and the number of conversation relationships (vertical and not horizontal relationships). This is shown on the following page.

Students should now be able to describe the pattern in the following manner:

If you multiply the number of persons in the relationship by 1 fewer and then divide the answer by 2, you will know the result for the number of conversation relationships. From here it should not be difficult to see that if there are 30 persons, then they will have 435 conversation relationships – that is, $(30 \times 29) \div 2$.

How to Create Original Problems

As was noted in the previous section, teachers should be able to create non-routine problems. We suggest some ways in which this can be done.

- Incorporate real-life situations into the problem-solving task. Mathematics is an integral part of everyday life. You could, therefore, seek to develop mathematical problems that are true to life. For example, you could write a problem that requires students to explore the different types of pastry and drink that are sold at the canteen/tuck shop (for example, three types of drink and five types of pastry). Have students determine the number of different types of drink and pastry combinations possible and then introduce a project in which students use their lunch time to see if the entire class could buy all the different combinations predicted.
- **Build problems around cognitive roots.** Problems are often developed using what is called a cognitive root an important concept, idea, principle or approach on which future understanding will be built. An example of a cognitive root is that of area. At its core, area refers to the number of unit squares that can be used to cover a region. This idea can be built into many problems, such as determining the greatest area possible for a rectangle with a perimeter of 12 units.
- **Transform routine problems into non-routine problems.** We should do the following when attempting to transform routine problems into non-routine problems.
 - 1. Create problems that allow students to explore concepts, not just apply algorithms. For example, by asking students to create pairs of fractions that when added give a sum of 1
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or less they may still apply the LCM algorithm – children will most likely use it to add the fractions they make up – but they are also being asked to do far more than just apply this algorithm. They have to:

- o consider the answer to see if it is more than 1;
- explore the role of denominators and numerators in determining the value of fractions; and
- consider the fact that some fractions simply cannot be used in the question, for example $\frac{4}{3}$ is not possible.
- 2. Create problems using general principles as a guide while ensuring that these guiding principles are not limited to algorithms. For example, LCM is built on the idea that separate events that recur frequently, but at different intervals, may occur a few times together. For example, if the fifth tile on a floor is triangular and the third tile is red, then the 15th tile should be both triangular **and** red. Many problems can be created using this as a guiding principle. Another such problem is:

If hot dog rolls come in bags of 8 and frankfurters sold in packs of 12, what is the least number of each pack that a person must buy in order to have 1 frankfurter for every roll? List other numbers that will ensure that a person has 1 frankfurter for every roll. In general, what numbers will ensure that a person has 1 frankfurter for every roll?

3. Open the task, by stating restrictions or characteristics that the answer should have. For example, a routine problem is $\frac{1}{4} + \frac{1}{3}$. In modifying this problem we could ask the students to:

use the following numbers to mak	ke pair	s of fra	ctions	that when added	
gives a sum of 1 or less:	1	2	3	4	6

The non-routine problems that the teacher creates should involve or enable the development of one or more of the following skills:

- Making and testing hypotheses
- Recording
- \circledast Selecting and using information
- Solution Identifying and extending patterns
- Making and proving conjecture
- Communicating
- Seneralizing

- Representing
 Classifying
- S Modelling
- Simplifying
- 🔊 Justifying
- Predicting

These problems should also encourage students to:

- seek solutions, not just memorize procedures;
- **explore patterns**, not just apply formulas;
- formulate conjectures, not just do exercises;
- consider solutions, not just produce answers;
- develop methods, not just follow steps;
- **test hypothesis**, not just perform algorithms and
- learn to **transfer information**, not just retain.

Extensions

Students are usually asked to solve an **extension** to a problem after they have solved the problem itself. An extension is an addition to a problem that asks the solver to apply what they have already learnt from solving the original problem to the modified version. Consider the following problem.

"If we were to write all the numbers from 1 to 1,000, how many times would we write the digit 9?"

Now, consider the following extension to the problem.

"If we wrote out the numbers 1 – 100,000, how many times would we have written the digit 9?"

This extension requires students to find an efficient way of counting the number of times 9 is written in each hundred, thousand and ten thousand and then extend this to 100,000. One does not expect them to actually write and count out all the 9s in the numbers from 1 - 100,000. Such an extension is useful for a child who was able to resolve the initial problem quickly and needs further challenge.

An extension is **not** simply a variation of a previous question in which a number has been changed – unless changing the number changes the problem entirely. Nor is it a variation of the tasks simply to create more work for the students and keep them working while others complete the task. The following would be a poor example of an extension to the problem given above.

Determine the number of times we would write the digit 3 if we attempted to write out the numbers from 1 - 1,000.

This is a poor extension since it is asking students to repeat the process they had carried out before requiring little or no new thinking.

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SECTION 3:

PROBLEM SOLVING – PROCESS AND STRATEGIES

The Problem-Solving Process

The problem-solving process is a set of steps that are sufficiently general so that they can be applied to any problem in order to obtain a solution. According to *Hattfield, Edwards, Bitter & Morrow* (2005), Polya identified four steps in the problem solving process.

- 1. Understand the problem
- 2. Devise a plan
- 3. Implement the plan
- 4. Look back

These steps are described in more detail below.

The Census Taker Problem (introduced before) will be used as an example to illustrate the activities that are involved in each step.

A census taker comes to the house of a mathematician and asks how many children he has and what their ages are. The mathematician replies that he has three children and the product of their ages is 72. The census taker replies that he has not been given enough information to determine their ages. The mathematician adds that the sum of their ages is the same as his gate number. The census taker leaves to check the gate number but returns shortly to say that he still does not have enough information. The mathematician thinks and says the oldest one likes chocolate ice cream. The census taker replies that he now has enough information and leaves. *What are the ages of the three children*?

Step 1 – Understand the problem

At this point in the process you should spend time decoding the problem to ensure that you are able to take from it the important facts and considerations. In doing so the following considerations will be necessary.

- You must understand exactly what the problem requires you to do. Let us examine the Census Taker Problem more closely, for example. It clearly asks you to find the ages of the three children. What is implied from the story is that their ages should sum to an unknown gate number and should multiply to give 72.
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- Familiarize yourself and pay attention to important words in the problem. While it is not absolutely necessary for obtaining a solution, you may need a bit of background information on what a 'census' is. Phrases such as 'product of their ages' 'sum of their ages' and 'the same as', however, need to be emphasized and understood in order to decode the problem.
- *Restate the problems in your own words*. After reading the Census Taker Problem, you should be able to identify that the mathematician has said 'the ages of my three children give a product of 72 and a sum that is the same as my gate number.' More importantly, you should be able to say that this information is not enough to solve the problem.
- If necessary, represent the problem using two- and/or three-dimensional models (tables, diagrams, flowcharts, physical models, etc.) The creative problem solver, for example, may decide to model the problem as follows:



- *Identify the important and the missing information in the problem*. You should appreciate, for example, that the census taker leaves thinking he has enough information. It was only *after* he checked that he realized that he still did not have enough information to determine their ages. Additionally, the fact that the oldest child liked chocolate ice cream was not as important a clue as the fact that there was an *oldest* child!
- Condense and compartmentalize the problem into simpler parts and tasks. Having read the problem, you should be able to identify that you need to find ways to make a product of 72 using three numbers. You should also know that there is something peculiar about the sum of the three numbers.

Step 2: Devise a plan

Taking into consideration all that has been unearthed from step 1 (understanding the problem), the next step is to devise a plan to solve the problem. This step involves choosing and creating strategies that you reasonably expect will lead to a solution. The solution plan is the bridge between what you know from reading the problem and what you are trying to find out. Usually, the plan you develop is informed by the information you have been given and that which you can infer. Developing a plan to solve a routine problem usually involves selecting the right algorithm or series of operations to apply. In attempting to solve non-routine problems, however, no known algorithm exists or is accessible to you; therefore, you are required to devise a plan for yourself.

This step is, therefore, very important as it is primarily what makes solving non-routine problems such an important activity in developing students' critical thinking skills. In considering The Census Taker Problem, for example, we can attempt to obtain a solution by trying the following strategy.

- Make a list of the various three number combinations that can be multiplied to give 72.
- Explore the sums of these numbers and note what is observed.

Step 3: Implement the plan

Once the plan has been developed, implementing it requires you to be deliberate, organised, accurate and observant. Each of these terms is explained below.

- Deliberate: Do not just write down results, try any case, select numbers randomly and make wild guesses. Be deliberate in how you implement the strategy. This will help you to see a solution and to minimise the chance of errors. For example, in attempting to solve the Census Taker Problem, we need to write down the different sets of three factors of 72, it may help us if we decide that in writing each combination, we will always start with the smallest factor. This helps us avoid writing 2 and 3 and 12 as one set of factors having already written 3 and 12 and 2.
- **Organised**: This complements being deliberate. If you have a system for organising your factors, you may stumble across information that you would not have seen otherwise. Tables and diagrams are very effective for this purpose.

Child 1 (age in years)	Child 2 (age in years)	Child 3 (age in years)	Sum
1	1	72	74
1	2	36	39
1	3	24	28
1	4	18	23
1	6	12	19
1	8	9	18
2	2	18	22
2	3	12	17
2	4	9	15
2	6	6	14
3	3	8	14
3	4	6	13

- In looking at the sets of factors of 72, and bearing in mind that the sum of these factors is likely to become important, it is perhaps best to organise the information as shown in the table above.
- Accurate: In implementing the solution, it is important that you follow the plan accurately. Devise systems for checking the accuracy of your work – not just for computational errors but also for omissions, repetitions, etc. In the table, above, for example, since 36 is a factor of 72 then all factors of 36 (1, 2, 3, 4, 6, 8, 12 and 36) must be accounted for.
- **Observant**: Being observant means that you will be able to identify interesting and important aspects of your results and make the connections between these results and the questions. Being observant, for example, allows us to see that there are two sets of numbers in the table that add to 14:

Observe, as well, that 14 is the only sum that is repeated. We can conclude, therefore, that the gate number must be 14 as, otherwise, the census taker would have been able to solve the problem. If the gate number were 17, for example, then the children's ages would be 2, 3 and 12. The census taker returned, therefore, because he realised that their ages could be either **3 and 3 and 8** or **2 and 6 and 6**. When he heard that there was an oldest child, therefore, he concluded that their ages must be 3 and 3 and 8.

Step 4: Look back

Having solved the problem, the importance of this step is to encourage reflection on the problem-solving process and to verify that your answers are valid. By reflecting on the process used to solve a problem, you take from it the lessons that would be useful when confronted with another problem of this nature. For the Census Taker Problem, depending on what mistakes you made, you may choose to reflect on:

- the importance of being organized in representing the information that will be used to solve the problem;
- the value of being deliberate in how the results were written down;
- the need to pay attention to the language used in describing a problem; and
- the value of just starting a problem by writing down, organising and exploring what you already know.

The verification of answers involves ensuring that they are logical and reasonable with reference to the context created in the problem. You may also want to explore your findings to see if general principles can be abstracted.

Problem-Solving Strategies

Students engage in problem solving when they develop, select and apply strategies to obtain solutions to tasks for which no immediate answer or solution path is known. As mentioned in the previous section, the selection and implementation of strategies are important aspects of the problem-solving process which students go through when they are confronted with non-routine tasks. While a typical, non-routine problem can be solved in a variety of ways and while no exhaustive list of the various problem-solving approaches and strategies can be made, in this section we will discuss some of the most popular effective strategies as well as how and when these strategies can be applied.

Logical Reasoning

Logical reasoning involves:

- examining the problem to determine what information has been provided;
- asking the right questions to determine what other information is needed; and
- using the information given as well as the answers obtained to make inferences and draw conclusions.

Logical reasoning will determine the sequence of steps, operations and/or other approaches, needed to find solutions to problems. Logical reasoning is an important tool in the mathematical problem-solving process and help students to develop a deep appreciation for what they are doing. The sample problem below outlines how logical reasoning can be used to solve problems.

		J	anuary 2011					
Sun	Mon	Tues	Wed	Thur	Fri	Sat		
						1		
2	3	4	5	б	7	8		
9	10	11	12	13	14	15		
16	17	18	19	20	21	22		
23	24	25	26	27	28	29		
30	31							
30	٦١							

What day of the week will be the 179th day immediately after a Thursday?

In examining the calendar, we see that seven days after Thursday take us to the next Thursday. This observation can lead to the following logical deductions.

- Starting from Thursday, any number of days being a multiple of 7 will take us to a Thursday.
- If 179 is a multiple of 7, then it must be a Thursday; otherwise, we can use the remainder to determine how far away from Thursday it is.
- So, instead of counting the days on the calendar, we can now divide 179 by 7. This yields a result of 25 groups of 7 days with a remainder of 4 days.
- Going back to the calendar and counting four days from Thursday gives us Monday, the desired response.

Guessing and Testing

Guessing and testing involves selecting likely solutions in accordance with the conditions in a problem and checking to see if these solutions are correct. This approach can save time if the options to a solution are few. On the flipside, however, it can be time consuming if there are too many options that must be eliminated. For example, in order to determine which two consecutive whole numbers will give a sum of 89, a guess could certainly save time. Consider, though, if you were asked to determine the number of handshakes that would occur in a room where there were 200 persons shaking each other's hands – it would be almost impossible to make appropriate and intelligent guesses.

Using this method of solving problems does not mean that the problem solver will simply shout random numbers until they come across the right ones. Each guess should be used to refine further guesses and to exclude or focus on particular numbers for future guesses. Let us explore this method using the consecutive whole numbers problem above. After a few guesses students should realize that:

- the numbers must be in the '40s' as any pair of numbers in the '30s' will give a sum that is more than 60, but less than 80. Furthermore, numbers in the 50s will give sums that are 100 or more; and
- at some point, consecutive numbers in the '40s' will give a sum that is in the '90s' all pairs in which 46–49 is present will give a number in the '90s'. For example, 47 and 48 give 95. These observations are shown below.



The consecutive numbers, therefore, fall between 40 and 45. At this point, the problem solver can easily identify the requisite numbers as 44 and 45.

Solve a Simpler Problem

This method in problem solving suggests that students take a known problem and make it simpler, without changing the focus of the problem. Consider the following problem.



• Solving a simpler problem, however, helps in determining the answer to this question. Consider the table below, which gives the result of 2¹ up to 2¹².

Problem	Power	Result	Last digit
2 ¹	1	2	2
2 ²	2	4	4
2 ³	3	8	8
2 ⁴	4	16	6
2 ⁵	5	32	2
2 ⁶	6	64	4
27	7	128	8
2 ⁸	8	256	6
2 ⁹	9	512	2
2 ¹⁰	10	1,024	4
2 ¹¹	11	2,048	8
2 ¹²	12	4,096	6

By looking at the table, you realize that:

- the last digits are simply a repetition of the numbers 2, 4, 8, 6.
- all the results for powers that are multiples of 4 (2⁴, 2⁸, 2¹², etc.) will have a last digit of 6. Since 200 (the power in 2²⁰⁰) is a multiple of 4, then the result will also have a last digit of 6.

Make a Table

Some problems are very complicated and ask students to work with numbers that are too large for them to manipulate. In these cases, students should be able to create a table and look for patterns that they can use to solve complex cases. When this method is used, two key activities are usually involved. These are:

- exploring the results of simpler cases and recording the answers in a table; and
- identifying a pattern from the table that can be extended over many or any number of cases.

For example, the problem highlighted below can be solved by using a table.

The active 4-H Clubs in each school in your parish are visiting each other. Each club will see each other twice – once at its own school and another time at the other club's school. How many visits will be

made if there are 50 active 4-H Clubs in your parish?

Now at first glance, the problem is likely to look daunting – no known method exists and it is difficult to make guesses when there are as many as 50 clubs, since no basis exists for making these guesses. However, if we try to simplify the problem by choosing to work with, say, four clubs (**A**, **B**, **C**, **D**) we may get further insights which we can apply in solving problems involving larger numbers. Now, importantly, we have to devise a system for recording how many visits are made with four clubs visiting each other twice.

Avoid randomly listing possible visits. The list below shows the danger of simply listing combinations (we are using ' $\mathbf{A} \Rightarrow \mathbf{B}$ ' to mean ' \mathbf{A} visits \mathbf{B} ', while ' $\mathbf{B} \Rightarrow \mathbf{A}$ ' means that ' \mathbf{B} visits \mathbf{A} ').

• A⇔B	• B⇔D	• A⇔D
• B⇔A	• C⇔A	• A⇔C
• C⇔A	• D⇔A	• C⇔D
• A⇔D	• B⇔A	• D⇔B

In the list above, some visits have been repeated while others have been omitted. Instead, try to be as systematic and organized in listing possible combinations.

In the system outlined below, all the visits that Club A makes are listed in the first row, those that Club B makes are listed in the second row and so on.

ΑvΒ	A v C	A v D
B v A	ΒvC	ΒvD
C v A	C v B	CvD
D v A	D v B	DvC

This system gives us an insight into how the problem can be solved when four clubs exist, each of them will make one visit to the three other schools giving us a total of 12 visits. An even more sophisticated system of **representation** is shown on the following page.

Now that we know that four clubs will make 12 visits, where do we go from here? Does it mean that the number of visits is three times the number of clubs? Will 99 clubs make 297 visits? Perhaps before we come to any conclusions, we should try a few more cases. Let us select carefully what number we try next. Since we know what happens when there are four clubs, let us explore the number of visits when there are 2, 3 or 5 clubs. This allows us to see any patterns that might present themselves.

	Α	B	С	D
A		ΑvΒ	ΑvC	ΑvD
B	ΒvΑ		ΒvС	ΒvD
С	C v A	СvВ		CνD
D	D v A	DνB	DνC	

The table below shows how many visits are made by up to five clubs.

Number of Clubs	2	3	4	5
Number of visits	2	6	12	20

Now that we have a table, we can look for horizontal (side to side) as well as vertical (top down) patterns and relationships. Write down the patterns that you observe. Specifically, look for:

- **differences** (1st and 2nd);
- relationships;
- rules; and
- patterns such as symmetry, odd-even, number types, etc.

Points to note

- Horizontal differences occur when we see relationships between numbers that are arranged side by side. These relationships help us to predict the number of visits by 'counting up' using the first difference.
- The first difference refers to the difference between adjoining numbers in a table.

Number of Clubs	2	3	4	5	6
Number of Visits	2	6	12	20	30
	+	4	+6 +	- 8	+ 10

In the table below the first difference increases by 2 each time.

We can, therefore, predict that if there are seven clubs there will be 12 more visits than when there were six clubs. We can also complete the following table.

No. of clubs	4	5	6	7	8	9	10	12	14	20	30	50
No. of visits	12	20	30	42								

Using this technique we can count up to 50 clubs to determine the number of visits made. Obviously, this would be a time-consuming process. It would, therefore, be better, if we could determine a relationship between the number of clubs and the number of visits – in this way we would be able to predict the number of visits as long as we knew the number of clubs.

This relationship is a **vertical** one (top-down) and is shown below:



To determine the number of visits for a particular number of clubs we would:

- subtract 1 from the number of clubs; and
- multiply this number by the number of clubs.

To determine the number of visits for seven clubs, therefore:

$7\times(7-1)=7\times6=42$

50 clubs would therefore make 2,450 visits. That is,

$50 \times (50 - 1) = 50 \times 49 = 2,450$

N.B. Based on the grade and performance levels of children, you may also discuss a general solution – if there were *n* clubs, then the number of visits will be n(n - 1).

Draw a Diagram

Drawing a diagram is a useful technique to apply in solving geometric problems. Let us look at this problem.

Tom is planting a garden of tomato plants. He can place seven plants so that they form five straight lines with three plants on each line. How is this possible? Here are two possible solutions.



In creating the diagram students may be allowed to use counters, putty or any suitable manipulative to aid the process. This is particularly useful when students have to make many attempts before a final answer is produced, or when a part of a drawing may have to be changed without necessarily starting over.

Making a Graph

Graphs allow trends to be shown and patterns to be extended. Usually, problems that can be solved using tables may also be solved using graphs. Consider the problem below, which gives us an idea of how graphs can be used in problem solving.

There are three triangles in the diagram below. The diagram has a total of seven lines.



Michelle is trying to figure out how many lines would be in diagrams which have from three up to nine triangles. So far she knows that a diagram with:

- five triangles would have 11 lines; and
- nine triangles would have 19 lines.

Help her to determine the number of lines for the diagrams that she has not drawn as yet.

Drawing a graph to represent the information known thus far helps (see graph). Even though the graph shows the number of lines for diagrams with 3, 5 and 9 triangles, we can also determine the number of lines in diagrams with 1 – 9 triangles. For example, by reading the graph we can deduce that a diagram having:

- 4 triangles will have 9 lines; and
- 7 triangles will have 15 lines, and so on.





Work Backwards

The process of reversibility is a key feature of mathematics. Reversibility involves starting at the end (usually from the response, which is already known) and working backwards in order to obtain missing information. Problems to which the **final outcome**, not the desired **response**, is known are suited for such a method. In working backwards any other previously mentioned method may be incorporated. The problem below shows how the process of reversibility can be used.

Joe, Nick and Tom are sharing peanuts. The following steps are followed:

- Joe gives Nick and Tom as many peanuts as each already has.
- Nick then gives Joe and Tom as many peanuts as each of them then has.
- Finally, Tom gives Nick and Joe as many peanuts as each has.

If at the end, each has sixteen peanuts, how many peanuts did each have at the beginning?

	Number of Peanuts				
	Joe	Tom	Nick		
Final amounts are (total 48)	16	16	16		
In the last step, Tom gave each of the other boys, the amount they had. Their peanuts were therefore doubled. The amounts they had BEFORE Tom shared with them were:	8 (Joe, therefore, got 8 peanuts from Tom)	32 (Tom gives away 16 peanuts – 8 to Nick and 8 to Joe)	8 (Nick got 8 peanuts from Tom)		
Before Tom shared, Nick gave the other two boys the same amounts they had before, hence he too doubled what each person had before. The amounts they had BEFORE he shared were:	4 (therefore, Joe got 4 peanuts from Nick)	16 (Tom got 16 peanuts from Nick)	28 (Nick gave away 20 peanuts – 16 to Tom and 4 to Joe)		
In the beginning, Joe gave to each of the other two boys the amount they already had. Hence, he too doubled their amount; but BEFORE he did they each had:	26 (he gave Joe 8 and Nick 14)	8 (he got 8 more from Joe)	14 (he got 14 more from Joe)		

The steps can be simplified and reversed as follows:

From the given deductions, at the beginning Joe had 26 peanuts, Tom had 8 peanuts and Nick had 14 peanuts.

Acting it out/using models

Some problems are action-based or require the observation of physical formations of people or objects in order to answer related questions. These problems are best acted out or modeled as they provide students with real experiences and a deep understanding of the problem. The handshake problem is a popular problem that is usually acted out. In this problem, a certain number of persons (let us say 15) are in a room and are shaking each other's hands. The question is to determine the number of handshakes that are exchanged. Putting students into groups and having them shake hands to solve the problem is ideal in this case.

SECTION 4 PROBLEM SOLVING IN THE CLASSROOM

Problem Solving as an Approach

In the Revised Primary Curriculum (RPC), problem solving is not treated as a topic or as a strand. Instead, teachers are encouraged to use problem solving as an approach to teaching the five content strands and to integrate it into all mathematics lessons.

Problem solving requires that students retrieve, transfer and apply previously learnt information to new or varying situations. Using problem solving as an approach means that students learn mathematics through real contexts, problems, situations and models. According to Van de Walle, Karp and Bay-Williams (2010, 32), "most, if not all important mathematics concepts and procedures can best be taught through problem-solving". Teaching through problem solving means that the questions teachers ask must be those that are open and that facilitate critical thinking. Additionally, the tasks that students are asked to perform must involve them in activities that require them to do more than apply algorithms. For example, consider the task below.

John has 4 birds and Michael has 6 birds. In total, how many birds do they have?



This task requires children to call upon and apply their addition algorithm. Consider the following variation to this question, however.

John has ____ birds. Michael has ____ birds. In total they have 9 birds. How many does each have?



In the second task, children are involved in a greater level of thinking as they are the ones who must think of and select addends to make the number sentence true.

Planning for Problem Solving

In planning for problem solving, a teacher must recognize and appreciate that these lessons should not be teacher centred. For this to take place, more emphasis should be placed on the students' thinking process and their active involvement in the learning process. You should prepare

students mentally for the task and be sure that the task is understood. Allow students time to grapple with problems, to work independently to find strategies and solutions on their own or in groups as well as learn to evaluate their own results. In order for students to do this successfully, they must develop most of the following habits outlined by *Moulds and Ragen* (n.d.).

- Persistence: Effective problem solvers are prepared to stick to a task until it is completed. They do not give up in despair as soon as they become unsure as to how to proceed, or if the strategy they have used so far has not yielded much success.
- 2) Managing Impulsivity: Students who are problem solvers are able to manage their impulses. Their actions are deliberate and thought out – they do not shout the first answer that comes to mind, write down the first thought that they have and take steps without considering options. In short, problem solvers think before they act and are able to justify their actions.
- **3) Communicating Clearly**: Problem solvers know that patterns, relationships and ideas are seen most clearly when an orderly system of expressing and recording these ideas has been developed and strictly adhered to. They say what they mean and are usually able to make subtle distinctions in describing their ideas, approaches and solutions.
- 4) Thinking Flexibly: Problem solvers are able to change their strategy when it does not meet with success, they are able to develop approaches that are variations of what they already know and are ready to accept other points of view. Problem solvers are prepared to accept ambiguity and confusion in their thoughts, knowing that these are areas of potential exploration.
- **5) Striving for Accuracy and Precision**: Students often lack the discipline to check their work to see if it is correct and/or complete. Problem solvers, however, know that omissions, deletions, imprecision and inaccuracies in speech and their systems of representation make problem solving difficult.
- 6) Applying Past Knowledge: Problem solvers do not see every problem as being new and for which a different approach or set of activities is required the lessons learnt from each problem solving experience are used to solve new problems. Students who become problem solvers must not have episodic experiences with problems each problem should not be presented as being a discrete experience with its lessons to be *retained* but not *transferred*.
- 7) Working Interdependently: Problem solvers know how to work as a part of a group; they know how to use the many ideas that exist within a group to determine a solution.

During the planning process for problem solving, teachers must consider how they will model and encourage the development of the above habits so that students will be guided in developing them.

Van de Walle, et al. (2010) suggest that in planning to teach via problem solving the following should be done:

1. Determine the mathematics and goals.

What is it that students should be able to do when this lesson is done?

2. Consider your students' needs.

Focus on the individual needs of your students, including learning styles, previous knowledge, learning gaps, and misconceptions which may exist.

3. Select, design, or adapt the tasks to be used.

- Carefully selected tasks will help to maximize students' learning outcomes and ensure that the lesson runs smoothly. To this end, *Ollerton* (2007) proposed the following:
 - o Align the problems to be used to the goals and needs outlined.
 - Ensure that tasks or problems posed engage students in thinking about and developing the important mathematics they need to know (Van de Walle et al, 2010) and the skills they need to have.
 - o Find starting points that all pupils can engage in the problem at the initial stage. This empowers them to overcome potential anxieties and aids them in developing confidence so that they are prepared to tackle the problem without giving up prematurely.
- Problems should be accessible and at the same time puzzling.
- Use more open-type questions.
- Ensure that problems are extendable this will enhance differentiated learning.
- Foster independent learning.

Organizing Students for Problem Solving

Burns (2000) postulates that in organizing one's class for problem solving, one of the first things to consider is how students will work – whether individually, in small groups or as a class. Having children work in small groups – say two to five members per group – is widely encouraged. The group members are organized to sit together at tables or desks which may be placed in clusters. This will make it possible for more active involvement among students and reduce the likelihood of students working on an individual basis. In this way, more students are afforded the chance of voicing their ideas and receiving feedback. It is highly recommended that students be selected randomly for groups and that the groups are changed around regularly to give students the chance of working with everyone, over time.

Having established the working groups, guide students in understanding that:

- each child is responsible for his/her own work and behaviour;
- each group member must be willing to help any group member who needs help; and
- the teacher should only be consulted when everyone in the group has the same question.

In order to determine how to organize students to maximize success in the mathematics classroom, teachers might need to pay particular attention to the following considerations.

Readiness: Within your classroom, you may find that some students are likely to display a high level of readiness for problem-solving tasks – their thinking strategies are in place, their problem-solving habits are well developed and they are able to work independently of you, the teacher. Others, however, may just be developing these skills and may approach problem solving with great apprehension. Any grouping of students for problem solving should, therefore, be organized in such a way to ensure that:

- groups contain students at different levels of readiness; and
- each child still has multiple opportunities to encounter and attempt non-routine problems on an individual basis so that proper problem-solving habits may be developed.

Interest/passion: Some students have a greater interest in and passion for problem solving than others. These students should be scattered across groups so that they can drive the attempts made by each group.

Gender: The culture that obtains in your classroom will help you to determine whether or not you will have groups made up of both genders. Bear in mind, however, that research has shown that males and females approach problem solving differently and that this is likely due to their cognitive functions as a result of biological differences (Zhu, 2007). While these differences do not necessarily imply superiority of one gender over another in terms of their problem-solving skills, it would be ideal if groups were made up of both genders so that these groups would benefit from the approach taken by each gender.

Learning profiles: Some persons are auditory learners – they learn best by hearing and discussing the information – while others are visual learners and need to see two- or threedimensional representations of the information they are attempting to process. In addition, some persons are tactile learners and need to physically interact with manipulatives and models in order to discover principles and concepts. These different learning profiles are to be distributed across the different groups that are formed. This will ensure that a diversity of approaches can be used by each group.

Roles: Depending on the task that students are attempting, a group may need a leader, a scribe, an artist, a materials manager and an organizer. Assigning students to groups, therefore, should take these various roles into account.

Dealing with wrong/incomplete approaches and/or answers

In order to effectively handle students' incorrect and/or incomplete responses, you must be aware of possible responses. You should be able to provide students with counter-examples or guide them in identifying same. In many instances, children have done the groundwork necessary to make inferences, observe patterns, draw conclusions, make predictions and produce results. In these instances, it is necessary for you to ask the right questions and provide the right prompts. You may ask probing questions such as:

- Have you noticed any pattern(s)?
- How can you describe that pattern?
- What do you think will happen next? How do you know?
- Have you checked to see if there is a common difference between consecutive numbers?
- Are the numbers you have obtained so far of a particular type are they all odd, even, prime, etc.?

Sometimes students are stuck at a particular point and are unable to think of an alternative strategy that may help them to resolve the problem. In such an event, you may want to highlight, from the attempts students have made so far, areas that could be explored in greater detail. Alternatively, you may suggest a strategy with carefully worded prompts such as "what if you had tried to..." or "do you think that only odd numbers should have been used to...?. **You should not solve the problem for students or tell them how to do so.**

At the end of a lesson, when the class discusses the solutions, you may find out that a student or a group has an incorrect or incomplete response. At this time, it is important that you allow the class to discuss this response by looking at elements of the approach that are correct, the student's or group's motivation for choosing the approach taken and whether or not it could be modified to give an acceptable response.

A variety of answers are likely to result from a group discussion of the responses to a problem, as students make their own observations and explore their own unique ideas and strategies. Some of these strategies may all lead to an acceptable answer and, hence, all of them should be regarded as being **effective**. The teacher, however, may guide students into identifying **the most efficient way of producing a result**.

SECTION 5

PROBLEM BANK

This section contains a bank of problems for students from grades 1 – 6. As you consider using these problems in your classrooms, bear the following in mind.

- Though each problem has a suggested grade level, you may find that your students are either able to access problems suggested for higher grades or are unable to access those for their grade levels. In associating the problems in this section with grade levels, the only considerations made were the objectives from the curriculum – you may find, however, that your students' functioning level and placement levels are different.
- Some problems may need to be modified to meet the specific needs and context of your students and your classroom – make sure you read through and attempt a problem before you ask students to solve it.
- No indication is given as to how each problem can be integrated into a lesson or at what point in the teaching sequence of a strand or concept a related problem can be introduced. Working through the problems related to a strand will help you in addressing this issue.
- Some problems have not been linked to any particular strand or concept. These problems are still excellent and are great for developing students' critical thinking skills.
- For each problem, a suggestion as to how students can be organized is given. While we feel that each suggestion is the most effective way of attempting that problem, you do not have to adhere to the suggestions if your local conditions require that they be changed or ignored.



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Problem 1 – Number Chain

Grade: 3 or above Strand: Number Concepts: Odd, even, multiplication, division, half Student Organization: Pairs Skill: Computing

A number chain is a string of numbers in which numbers beside each other are related in a specific way. Rules are usually created in order to form a number chain. In this number chain, the following rules are to be observed.

- a) Start with any whole number.
- b) If you start with an even number, divide by two.
- c) If you start with an odd number, multiply that number by three and add one.
- d) Repeat this process on the answers that you obtain to form a chain of numbers.

The number chain below was created using 9 as the starting number and has 7 numbers.



- 1. Based on the rules above, create a number chain using 8 as your starting number. Make sure your number chain has at least 10 numbers.
- 2. What do you observe about the numbers in the chain?
- 3. Explore a few more starting numbers some odd and others even and see if you always observe this pattern.
- 4. List some starting numbers that will ensure that, except for 1, no odd number will ever be in the chain.

Extension

- 1. What if **the number were even** and you **still divided by 2**, then multiplied any odd number by **5 and added 1**. Would the same pattern occur?
- 2. Suppose you multiplied odd numbers by **7 and then added 1**, would this create the same pattern?
- 3. By what other numbers could you multiply odd numbers in order to create the same pattern?
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Problem 2 – Cutting Squares

Grades: 5 – 6 **Strands**: Measurement and geometry **Concepts**: Perimeter, square **Student Organization**: Small groups of 5 – 6 **Skill**: Predicting

Below is a 4×4 grid. Its perimeter is 16 units.



We will now remove a square with the length of each side being one unit less than the original square. From the 4×4 square, then, we will remove a 3×3 square. This is shown below:



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The 3×3 square is then reattached to the other side of what remains of the original square (shown below).



- 1. What is the perimeter of this new figure?
- 2. What would the perimeter be if we had started with:
 - a. $a 5 \times 5$ square?
 - b. $a 6 \times 6$ square?
 - c. a 7×7 square?
 - d. an $n \times n$ square?

Extension

Now, take out squares with sides 2 units less than the original and determine the perimeter if we had started with:

- a. $a 3 \times 3$ square
- b. $a 4 \times 4$ square
- c. a 5×5 square
- d. an $n \times n$ square

Problem 3 – Taking Counters

Grade: 2 or above Strand: Number Concepts: Factors, multiple, odd, even, less than, more than Student Organization: Pairs Skill: Strategizing

This is a game for two players. Start with a pile of 12 counters on the table. Each player, alternately, takes any number of counters from the pile. However there are two rules.

- a) The first player **cannot** take all the counters from the pile
- b) A player should not take **more than** twice the last number of counters taken.

The winner is the player who takes the last counter.

- 1. Play a few times with 10 counters and find a winning strategy to ensure that you always win, regardless of what your opponents do.
- 2. Does it matter who plays first?
- 3. What is the best strategy for each player?
- 4. Does the winning strategy work regardless of the number of counters?
- 5. Explore what happens in the game with different numbers of counters.

Extension

- 1. Suppose the person who takes the last counter loses?
- 2. Suppose a player has to take at least twice the previous number taken?

Problem 4 – Perfect Squares

Grade: 6 Strands: Algebra, number Concepts: Place value, squared numbers Student Organization: Small groups of 5 – 6 Skills: Predicting, generalizing, computing

Numbers which end in 5 (such as 15, 25, 35, etc.) have a particular appearance when they are squared.

Timz is squaring numbers ending in 5 and so far has the following results

Numbers	5	15	25	35	45	55	65	95
Result when Squared	25	2 25	<mark>6</mark> 25	12 25				

- a. Explain what is peculiar about the **last 2 digits** in the result of each squared number
- b. Describe the relationship between the number being squared and the digits highlighted in red in each result.
- c. Use these observations to complete the table.

Extension

1. Explore ways of predicting the answer when squaring numbers that end in 1 (such as 1, 11, 21, 31, etc.).

Problem 5 – A Tricky River Crossing Puzzle

Grade: 4 or above Student Organization: Individual or pairs Skill: Strategizing

There was once a showman travelling the countryside on a tour with his wolf, a goat and a cabbage. He comes to a river bank and the only means of crossing is a small boat which can hold him with only ONE of his belongings – the wolf, the goat or the cabbage.

Unfortunately, he dares not leave the wolf alone with the goat for the wolf would eat the goat. Also, he cannot leave the goat and the cabbage alone for the goat would eat the cabbage. Thinking carefully the showman realised that he could use the boat to transport himself and all his belongings safely across the river! How did he do it?

Extension

1. Would he still be able to transport himself and his belongings safely across if he also had a lion, which would eat either of the other two animals but not the cabbage? If so, how?



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Problem 6 – Map Folding

Grade: 4 or above
Strand: Geometry and measurement
Concepts: Area, square, rectangle, length, width, fraction
Student Organization: Small groups of 5 – 6
Skill: Strategizing

Below is a rectangular strip of paper the length of which is two times its width. It has been divided into 8 smaller rectangles, each one being $\frac{1}{8}$ th the size of the large rectangle.

1	8	7	4
2	3	6	5

Create a cut out of a similar rectangle:

- 1. Make its length 32 cm.
- 2. Make its width 16 cm.
- 3. Create small squares from the rectangle. Each of them should have sides of 8 cm in length.
- 4. Label the squares as shown above.

There are many ways to fold the large $32 \text{ cm} \times 16 \text{ cm}$ rectangle into a square that is one-eighth the size of the original figure. For example, in the rectangle below, the rectangle is folded and square number '1' is on top:



It can be unfolded to give on one side:



And, on the other side:



Note that '1' is on the other side of '2' and '4' is behind '5'. If we label the folds based on the order in which the numbers come next to each other in the folded paper, we obtain:

1 »» 8 »» 7 »» 4 »» 5 »» 6 »» 3 »» 2

- 1. Determine the number of different ways you can fold the rectangle into one-eighth of its original size
- 2. Record the folds using the method outlined above.
- 3. Fold the rectangle so that the numbers fall in the order 1 »» 2 »» 3 »» 4 »» 5 »» 6 »» 7 »» 8.

Extension

1. What if the rectangle were made up of squares numbered in the following manner:

1	10	9	4	5
2	3	8	7	6

2. Would you be able to fold the rectangle to get counting order?

Problem 7 – Packing Tray

Strand: Number Grade: 4 Concepts: Multiples, factors Student Organization: Individual or pairs Skill: Computing

A farmer is packing eggs in a tray.



If she takes up two eggs each time, she is left with one egg at the end; if she tries to take up three at a time, she still has one left over. The same thing happens when she tries to take up four, five or six at a time – on each occasion, she has one egg left over. However, if she takes up seven eggs at a time, she has no egg left over.

- 1. How many eggs does she have?
- 2. What other numbers are possible?
- 3. In general, what sequence of numbers is possible?

Problem 8 – Bicycles and Tricycles

Grade: 4 or above Strand: Number Concepts: Division, factors, multiples Student Organization: Pairs Skill: Computing

You rode your bicycle to the park and saw a number of other persons there. Some were riding bicycles and some were riding tricycles.



In total, you saw 51 wheels.

- 1. How many bicycles and how many tricycles were in the park?
- 2. What possible combinations of bicycles and tricycles in the park?
- 3. Explain why the number of bicycles in the park could not be equal to the number of tricycles.

Extension

- 1. If the sum of wheels on the bicycles and tricycles is odd, how can you determine the possible combinations of bicycles and tricycles?
- 2. How can you determine the combinations of bicycles and tricycles if the sum was even?

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Problem 9 – All in a Circle

Grade: 5 or 6
Strand: Geometry
Concepts: Angle, circle, line
Student Organization: Groups of 12
Skill: Predicting, generalizing, modelling

A number of students are standing in a circle as modelled below:



Each student stands so that there is always someone directly across from him or her – A is across from D, B is across from E and C is across from F.

- 1. In a circle with 12 persons, who stands across from the:
 - a) 4th person?
 - b) 7th person?
- 2. In another circle, the 9th person is across from the 34th person.
 - a) How many persons are in the circle?
 - b) Who is across from the 15th person in this circle?

Extension

- 1. Odd number positions appear to be across from even number positions. Is this always the case? Why or why not?
- 2. Suppose persons who face each other add their position numbers for example, if the 3rd person faces the 8th person, 3 and 8 are added to obtain 11.
- 3. What is the highest sum when there are:
 - a) 8 persons in the circle?
 - b) 20 persons in the circle?
 - c) *n* persons in the circle?
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Problem 10 – Filling the Pool

Grade: 5 or 6 Strand: Number Concepts: Volume, fractions Student Organization: Individuals or pairs Skill: Computing

A pool can be filled using any of four pipes. The first pipe can fill the pool in two hours, the second can fill it in three hours, the third in four hours and the last one in six hours.

1. How long will it take to fill the pool using all four pipes running together?

Extension

1. What if the pool had a leak which caused it to lose 10% of the water that went in it each minute. How long would it then take to be filled by the four pipes?



Problem 11 – The 400-Problem

Grade: 6

Strands: Algebra and number
Concepts: Subtraction, digit, number, place value
Student Organization: Small groups of 5 – 6
Skill: Reasoning, proving



Above are two subtraction problems. In each problem, a three-digit number is being subtracted from another one. In each problem, two **digits** of one number are represented by the letters *a* and *b*.

- 1. If the answer to problem 1 is the same as the answer to problem 2, what are the values of *a* and *b*?
- 2. Verify that these values are correct.
- 3. What values would *a* and *b* take if the problem were:
 - a) a 500-problem (4 is replaced with 5)?
 - b) a 600 problem (4 is replaced with 6)?
- 4. In general, what would be the values of *a* and *b* for an *n* hundred problem?

Extension

- 1. Explore what happens for a
 - a) 4,000 problem
 - b) 40,000 problem
- 2. For what types of problems will solutions always exist?
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Problem 12 – The Knight's Dance

Grade: 3 or above Strands: Geometry Concepts: Horizontal, vertical Student Organization: Small groups of 5 – 6 Skill: Strategizing

Knights are pieces used to play chess. They move in an L shape fashion as shown below.



From it current position the knight can be moved to any of the squares marked by dots by using L-shaped moves (verify this for yourself).

On a special 3 x 3 game board, two white knights and two black knights have been placed in the corners as shown below.



- 1. What is the fewest number of moves required for the black knights and the white knights to switch places (use different colour counters on a 3×3 grid sheet to help if necessary)?
- 2. What would the minimum number of moves be if it were a:

a) 4×4 square?

b) 5×5 square?

3. What would be the minimum number of moves if it were an $n \ge n$ square?
Problem 13 – Trading Places

Grade: 5 or 6 Strand: Number Concepts: Place value, addition Student Organization: Small groups of 5 – 6 Skills: Computing, reasoning

The following are two addition sums in disguise – each letter is being used to represent a digit in a number.

	Problem 1					Problem 2					
	F	5	Ν				R	5	I		Ν
+	Μ	5	N		_+	.	N	0	C)	Ν
	5	Ι	R		N	1	0	Т	С)	R

In the examples above, A is equal to 5 and N is less than 5.

- 1. Determine the values for each letter used in the problem.
- 2. Using the numbers you know so far and assigning letters to those unknown, make up a problem (you can use any operation) in which all words used are recognized in the English language.



Problem 14 – Change it Out

Grades: 1 – 2 Strand: Number Concepts: Addition Student Organization: Small groups of 3 – 4 Skills: Reasoning, computation

Toni has seven Jamaican coins that total \$38. From these coins, she cannot give Peta exact change for \$5 and she cannot give Clara exact change for \$10. However, she can give Kerrina exact change for \$20. What are the denominations of the coins that Toni has?

Problem 15 – Change it Out 2

Grade: 1 Strand: Number Concepts: Addition Student Organization: Small groups of 3 – 4 Skills: Reasoning, computation

Toni has \$20. Find as many ways as possible that she exchange them for other coins that total \$20. What is the fewest or greatest number of coins she can get for her \$20?



Problem 16 – Folded Corners

Grade: 2 Strand: Geometry Concepts: Shape, corner Student Organization: Small groups of 5 – 6 Skills: Predicting, generalizing

Make a cut out of a shape similar to below left - each corner has a letter.



In your cut out, fold over one edge (as shown by the dotted line) to produce a figure with 4 corners (shown and labelled above right).

- 1. Cut out a shape with 4 corners
- 2. Fold over one corner and count the number of corners that it has now.
- 3. Assume a cut had 5 corners and one is folded over, how many corners would it have then?
- 4. Figure out the same thing for cut outs with:
 - a) 6 sides
 - b) 7 sides
- 5. For any figure, how many corners would it have if one corner was folded over?

Extension

1. What if two corners were folded instead of one, how many corners would a figure end up with?

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Problem 17 – Mathematics at the Table

Grade: 4 or above Strand: Number Concepts: Addition Student Organization: Small groups of 5 – 6 Skills: Predict, generalise

In the school cafeteria, four people can sit together at one table.



However, if two tables are joined, then six people can sit together.



- 1. How many persons must be placed together in a row to seat:
 - a. 10 persons?
 - b. 20 persons?
 - c. 503 persons?
- 2. If the tables were arranged in a row, how many persons could be seated using:
 - a. 10 tables?
 - b. 15 tables?
 - c. 120 tables?
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Problem 18 – Counting Rectangles

Grade: 4 or above Strand: Geometry Concept: Rectangle Student Organization: Small groups of 5 – 6 Skills: Generalizing, predicting

The figure below, labeled A, is a rectangle.



Another rectangle is added; it is labeled B and is shaded.

Shape 2 »»



Shape 2 has a total of three rectangles, made up as follows:



Shape 3 is as follows:



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It has a total of six rectangles, as follows:



- 1. Draw the next shape (shape 4) and determine the total number of number of rectangles that it contains.
- 2. Predict the number of rectangles that will be in the 5th shape.
- 3. Explain how you made your prediction.
- 4. Complete the table below for the first 12 shapes.

Shapes	Number of rectangles
Shape 1	1
Shape 2	3
Shape 3	6
Shape 4	
Shape 5	
Shape 6	
Shape 7	
Shape 8	
Shape 9	
Shape 10	
Shape 11	
Shape 12	

Extension

- 1. How many rectangles are in the 100th shape?
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Problem 19 – Box it Up

Grade: 3 or above Strand: Number Concepts: Division Student Organization: Individual Skills: Reasoning, proving

This is a picture of the box in which Kerri stores her crayons.



Each box holds the same number of crayons and the crayons are alike in every way except for colour. Kerri realises that:



- 1. How many crayons does Kerri have all together?
- 2. How many crayons are there in each box?

Problem 20 – Missing Pages

Grade: 3 or above Strand: Number Concepts: Double, half, one more than, one less than Student Organization: Pairs Skills: Justifying, testing, proving

John opened his mathematics book and realized that a rat had eaten through some of the pages; as a result, page 44 is followed by page 51.



- 1. How many leaves (sheets of paper) were missing from the book?
- 2. What if page 44 were followed by page 57, how many sheets of paper would be missing from the book?
- 3. Now, suppose that the rat had also been eating the leaves of another book and in this case page 63 was followed by page 368, how many pages would be missing from that book?
- 4. Write out a general rule for finding the number of pages missing in any book.

Problem 21 – T-Patterns

Grades: 5 – 6 **Strand**: Algebra **Concepts**: Horizontal, vertical **Student Organization**: Small groups of 5 – 6 **Skills**: Generalizing, predicting

The figures below show the first three occurrences of a growth pattern made up of dots arranged horizontally and vertically. Each pattern is named based on the position that it occupies – the first pattern is called 'Pattern 1', the second pattern is called 'Pattern 2' and so on.



- 1. How many dots would there be in the 65th pattern?
- 2. Write algebraic expressions to represent:
 - a) the solution to the number of dots in any given pattern; and
 - b) any other relationships that you observe in the dot patterns.
- 3. A pattern has 218 dots. Which pattern is it?

Extension

- 1. Why do odd pattern numbers have an odd number of dots?
- 2. Suppose, instead of forming a T, the pattern formed a 'rotated H', as below; how would the relationship between pattern and number of dots change?



Problem 22 – Exploring Polygons

Grade: 5 – 6
Strands: Geometry, algebra
Concepts: Polygon, triangle, line, vertex,
Student Organization: Small groups of 5 – 6
Skills: Generalizing, predicting

Figure ABCDE, below, shows a pentagon.



From any vertex, only two **straight** lines that **pass through** the pentagon can be drawn to the other vertices, as shown in the pentagon below.



These two lines divide the pentagon into three triangles – ADE, ADC and ABC (above).

- 1. Assume that the polygon had six sides.
 - a) How many lines could be drawn through it?
 - b) How many triangles could be formed out of it?
- 2. In general, for any polygon, how many lines can be drawn through it and how many triangles will it have? Write algebraic expressions to represent these observations.
- 3. A polygon has 56 triangles.
 - a) How many sides does it have?
 - b) How many lines can be drawn through it?
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Problem 23 – Last One Standing

Grade: 6 Student Organization: Small groups of 6 – 8 Strand: Number, algebra Concepts: Halving Skills: Modelling, hypothesizing, generalizing, predicting

Five persons in a room decide to play the game 'Last man standing' which is explained below.

Standing in a circle and starting at any point, they alternately say 'in' and 'out' starting with 'in' (that is, the first person in the circle says 'in', the second person says 'out', the third person says 'in' and so on. Those who say 'out' fall out of the game by sitting). They continue to say 'in' and 'out' always maintaining that order until only one person remains standing. The last one to remain standing wins.

- 1. Verify that the last one standing when five persons are in the room is the third person (in the circle).
- 2. Who will win if there are 64 persons in the room?
- 3. Can you predict when the 30th person in the circle will always win?



Problem 24 – The 24 Challenge

Grade: 4 or above Strand: Number Concepts: Multiplication, division, addition, subtraction, fractions Student Organization: Individual or pairs Skill: Computing

In each box, below, there are four numbers. Use each number **ONCE** along with the four basic operations (+, \div , \times and –) **to make 24**. **ALL** numbers in the box must be used. There are no restrictions regarding how you use the four basic operations – you may choose to use all or any number of operations as often as you wish.

An example is done for you below.



Solution:

 $(5-1) \times (4+2) = 24$















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Problem 25 – Three in a Hole

Grade: 6 Strand: Geometry Concepts: Square, circle Student Organization: Pairs Skills: Strategizing, reasoning

Below is a strip of metal that is a cover for a drum. At any time, two holes will be opened and one will be blocked by a piece of wood. This piece of wood is to be cut so that it can block any of the three holes at any time. Only one piece of wood is to be used and it must be cut in such a way that it can fit **exactly** into any of the three holes to block it.



- 1. Describe how you would you cut the piece of wood. Consider the following:
 - a) Which design would you cut first?
 - b) Which design needs to be at the top of the wood?
 - c) Through which hole can each of the others always pass?

Problem 26 – Noughts and Crosses

Grade: 4 or above Strand: Geometry Concepts: Square, vertex Student Organization: Pairs Skill: Strategizing

The game 'Noughts and Crosses' is played on a 6×6 grid. Two players take turns to mark a 'nought' or a 'cross' on the square paper. The winner is the first player to make four marks that can be joined to form a square – each of these marks must be the vertex of a square. A sample game is shown below.



In this game, the player marking nought is the winner.

- 1. Play against an opponent and see if you can find a winning strategy.
- 2. Does the winning strategy depend on the size of the game board?
- 3. Instead of finishing the game when the first person has made a square, continue playing and determine how many squares each person can make.

Extension

- 1. What is the greatest number of squares that can be formed from a:
 - a) 4×4 grid?
 - b) 5×5 grid?
 - c) 6×6 grid?
 - d) $n \times n$ grid?

Problem 27 – A Weighing Problem

Grade: 6 Strand: Measurement Concept: Mass/weight Student Organization: Small groups of 5 – 6 Skills: Computing, strategizing

A shopkeeper has a balance scale and four weights. The weights are such that with them he could correctly weigh any whole number of kilograms from 1 to 40.

- 1. How heavy is each weight?
- 2. How could he manage to weigh all the different weights up to 40 kg?

Extension

- 1. What is the least number of weights that you would need to weigh any whole number of kilograms from 1 to 60?
- 2. What are these weights?



Problem 28 – The Triangular Garden Plot Grade: 6 Strand: Number Concepts: Squared numbers, triangle, ordinal numbers Student Organization: Small groups of 5 – 6 Skills: Hypothesizing, predicting, generalizing

Leslie has a **triangular** garden plot which has been divided into many rows. When planting corns, he plants one corn in the first row, three in the second row and five in the third row and so on.

- 1. How many corn seeds does Leslie plant in the twelfth row?
- 2. How many corn seeds does Leslie plant in the first twelve rows?
- 3. How many corn seeds does he plant in the *n*th row?
- 4. After the *n*th row, how many corn seeds would he have planted?

Extension

1. If he had planted the corns in the same way, except that he planted them in the 2nd, 4th, 6th rows and so on, would the answers for parts (c) and (d) above be the same?



Problem 29 – Frog Hopping

Grade: 4 or above Strand: Number Concepts: Fraction, half Student Organization: Small groups of 5 – 6 Skills: Computing, predicting

Fergus is a frog who wants to cross a river that is 64 units wide.

He decided to jump across the river but he only got half way across and landed on **stone A**. He then decided to turn back and jump the way he had come from, but, again, only jumped half the distance from where he was coming and this time landed on **stone B**. He kept on turning around each time only managing to get to half the distance he needed to go.

- 2. Will he ever land on a stone that he had landed on before? Why or why not?
- 3. How many units away from the starting point will Fergus be when he is on Stone E?
- 4. Will Fergus ever stop jumping and if so, where will he end up?
- 5. What if the fraction were three quarters instead of a half, where would he end up?

Extension

- 1. Can you make Fergus end up wherever you want?
- 2. What if the fraction varies after each jump?



Problem 30 – Chess Board Squares

Grades: 5 – 6 Strand: Number Strand: Geometry Student Organization: Small groups of 5 – 6 Skills: Predicting, generalizing, hypothesizing

Below is a 2×2 square.



In total, it has five squares of different sizes. These are shown below.





$\underline{ONE} \ 2 \times 2 \ SQUARE$



1. How many squares are there in a:

a) 3×3 square?

b) 4×4 square?

c) 8×8 square?

2. In general, how many squares are there in an $n \times n$ square?

Extension

- 1. What if we were looking for rectangles (a combination of two or more squares) instead, how many rectangles would be in a:
 - a) 3×3 square?
 - b) 4×4 square?
 - c) $n \times n$ square?

Problem 31 – Addend Pairs

Grade: 4 or above Strand: Number Concepts: Addition, addend Student Organization: Pairs/Small groups of 5 – 6 Skills: Computing, predicting, generalizing

Without repeating a pair, there are four pairs of numbers that add up to 6. These are: (0 and 6), (1 and 5), (2 and 4), (3 and 3)

Note that (6 and 0) is the same as (0 and 6) and so it is counted once.

- 1. Determine how many pairs of numbers will add up to:
 - a) 7
 - b) 8
 - c) 15
 - d) 1,001
- 2. Write a rule for finding the pairs of addends for an
 - a) odd number
 - b) even number

Problem 32 – Newspaper Mathematics

Grade: 3 or above Strand: Number Student Organization: Pairs Skills: Predicting, computing, reasoning



The picture above shows a sheet of newspaper. In it, page 9 is across from page 34. There are other missing sheets that fit between pages 9 and 34 – each of these missing sheets also has two pages as shown above.

- 1. How many pages are in the newspaper?
- 2. What pages are in the middle of the newspaper?
- 3. In another newspaper, page 7 is across from page 26. How many pages are in that newspaper and what would be the answers for parts (a) and (b) above?
- 4. Explain how you would be able to predict the number of pages in any newspaper, once you know the pages that are across from each other?
- 5. A newspaper has 50 pages. Which page is across from page 5?

Extension

- 1. Suppose, the newspaper shown in the picture above had a single page inserted in the middle. How many pages would the newspaper now have?
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Problem 33 – Chicken Combos

Grades: 5 – 6 **Strand**: Number **Student Organization**: Small groups of 5 – 6 **Skills**: Generalizing, predicting

A popular fast food outlet sells five different parts of a chicken – wings, legs, breast, rib and thigh. The restaurant offers a Big Meal Combo which contains two pieces of chicken. The restaurant does not allow customers to buy two of the same parts (such as two wings) in any one order.

- 1. How many different combinations of chicken can the restaurant offer in its Big Meal Combo?
- 2. By cutting pieces smaller, the restaurant now sells six different parts of the chicken. How many different combinations of chicken can it sell?
- 3. How many different combinations are possible if the restaurant sells *n* number of different parts of the chicken?

Extension

- 1. Suppose the restaurant allows customers to buy two pieces of the same kind, how would your answers change?
- 2. The restaurant also sells a Value Combo which has three pieces of chicken. How many combinations would be possible if the chicken were cut into five different parts?



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Problem 34 – Fractions of a Square

Grade: 4 or above Strands: Number, geometry Concepts: Fraction, square, area Student Organization: Individual/Pairs Skills: Estimating, justifying, spatial reasoning

The large outer square, PQRS, below represents one whole unit. It has been partitioned into pieces. Each piece is identified by a letter.



- 1. Decide what fraction each piece is in relation to the whole square and write that fraction on the shape.
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- 2. Explain how you know the fractional name for each of the following pieces:
 - a) A
 - b) C
 - c) D
 - b) F
- 3. Identify a piece or a collection of pieces from the square that will give you an amount close to:
 - a) $\frac{1}{5}$ b) $\frac{2}{3}$
- 4. Design your own fraction square. Make sure you include a:
 - a) $\frac{1}{5}$ b) $\frac{1}{3}$



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Problem 35 – Match Drawn

Grade: 6 **Strand**: Number **Student Organization**: Small groups of 5 – 6 **Skills**: Predicting, generalizing

In a football match between Jamaica and Canada, the final score was 2–2 (a draw). Though you did not attend the match, you reason that at half time the score could have been any score less than 2 – 2. Note that a score of 1 – 2 is different from a score of 2 – 1 as the home team's score is usually written first.

- 1. How many possible half time scores were there for a match that ended 2 2?
- 2. What about a game that ended 3 3?
- 3. How many half time scores are there for a match that ended n n?

Extension

1. Explore the number of half time scores possible for a match in which one team beats the other by a difference of 1 goal – such as a score of 2 – 1, or 3 – 2 (winning score is always written first).



Problem 36 – Measuring Jugs

Grades: 5 and 6 Strand: Number Concepts: Addition, subtraction Student Organization: Individual/Pairs Skills: Computation, reasoning

In each question below, there are three jugs – A, B and C. Your task is to find the most efficient way of measuring out a given quantity of water using these jugs. For each question, the size of the jugs as well as the volume of water you want to measure will change. *In some cases, you may have to use a jug more than once and in other cases not at all.* An example is done for you.

Example:

Measure out 27 units from the following jugs. Jug A holds 40 units Jug B holds 10 units Jug C holds 33 units

Answer:

(Jug A + Jug B) - Jug C = (40 + 10) - 33 = 27

1. Measure out 55 units from the following jugs.

Jug A holds 10 units Jug B holds 63 units Jug C holds 2 units

- Measure out 52 units if: Jug A holds 64 units Jug holds 10 units Jug C holds 2 units
- Measure out 14 units if: Jug A holds 100 units Jug B holds 124 units Jug C holds 5 units

 Measure out 20 units if: Jug A holds 23 units Jug B holds 49 units Jug C holds 3 units

Extension

1. Choose three jugs and make up a question for someone else to solve. Try to make it difficult for the other person to solve it.

Problem 37 – The Legs Problems

Grade: 2 Strand: Number Concepts: Addition Student Organization: Individual Skill: Computing

An animal can have up to 8 legs:

- Birds, etc. 2 legs
- Dogs, etc. 4 legs
- Insects, etc. 6 legs
- Spiders, etc. 8 legs







Problem 38 – The Trapezium Problem Grade: 6 Strand: Geometry Concepts: Trapezium, height, length, triangle, area Student Organization: Small groups of 5 – 6 Skills: Computing, hypothesizing, predicting

The diagram below shows a trapezium drawn on triangular lattice or isometric paper.



The trapezium contains 27 of the unit triangles. The dimensions of the trapezium are:

- AB (Top length) = 3 units
- CD (Bottom length) = 6 units
- AD or BC (Slant lengths) = 3 units

1. Write down the dimensions of each trapezium below. How many unit triangles does each contain?



- 2. In general, explain the relationship between the dimensions of a trapezium and the number of unit triangles that it contains.
- 3. How many unit triangles are in a trapezium where:
 - a) Top length = 4; Bottom length = 5; Slant length = 1
 - b) Top length = 1; bottom length = 3 slant length = 2

Extension

1. Given what you have done so far, explain how you would find the area in squared units for trapeziums of any dimensions.

Problem 39 – Karen's Troubles

Grade: 5 or 6 Strand: Geometry Concepts: Area, perimeter Student Organization: Small groups of 5 – 6 Skills: Reasoning, justifying

Karen is calculating the perimeter and area of rectangles. She calculates that the rectangle below has an area of 24cm² and a perimeter of 20 cm.



- 1. Verify that her calculations are correct.
- 2. The next question she sees (ABCD, below) has length and width that are *twice* as long as PQRS. She reasons, therefore, that its area and perimeter must each be twice as long as above. Explain to her why she is not correct.



3. Is it possible to find a pair of length and width that would allow ABCD to have a perimeter and an area that are each twice as much as PQRS? Is this true for **ALL** rectangles or just the ones Karen has seen?

Problem 40 – The Orange Juice Dilemma Grade: 5 or 6

Strand: Number Concepts: Fraction, division Student Organization: Individual Skills: Reasoning, justifying

Timz is having a party tomorrow. He has already made all the plans – he has already invited the guests and ordered the volume of orange juice he thinks can feed them.

He is having second thoughts now and wants to either change the number of persons attending **or** the volume of orange juice he provides **or** both. Complete the table below by indicating whether **each** person will get 'more', '*less'* or the 'same' amount of orange juice in each case. Is there any case for which you are not sure? Why?

		Amount of Orange Juice					
		Increase	Decrease	No chanae			
		mcreuse	Decreuse	No change			
	Increase						
No. of Guests	Decrease						
	No Change						

Problem 41 – Stand Out

Grade: 3 or above Strand: Number Concepts: Odd, even, prime, squared number Student Organization: Individual Skill: Justifying

From the following list of numbers, choose a number which is different from all of the others in the list. Explain why the number you have selected is different from the others.



Problem 42 – Jumping Frogs

Grade: 5 or 6 Strand: Number Concepts: Addition, subtraction, multiples, factors Student Organization: Individual/pairs Skill: Computing

Sammo the Frog has injured his legs and wants to visit the doctor's office. The doctor's office is five pads away from where he lives. This is shown below.



Usually, he would be able to hop from one pad to the next. However, because he has injured his legs he is having problems hopping – if he hops with his right leg he lands 11 pads away and if he hops with his left leg, he lands seven pads away.

- 1. Outline a series of hops that would ensure that he gets to the doctor's office.
- 2. Are there any other ways for him to get to the doctor from his home?
- 3. In general, on which pads could the doctor be located for the injured Sammo to get to him?

Extension

1. Suppose the pads are arranged in a circle and Sammo can only go in one direction on the circle; would he be able to get to the doctor's office?

Problem 43 – Mathematics Maze

Grade: 1 Strand: Number Concepts: Number Student Organization: Individual Skill: Strategizing

Below is a mathematics maze. Start at 1 and draw lines to connect numbers that are in counting order from 1 to 10 (include another variation – replace 2 in the corner with 9, replace 3 beside 8 with 10).

1 START	4	9	8	9
2	5	6	7	3
3	4	3	8	9
6	5	2	7	10 END

Problem 44 – Addition Maze

Grade: 1 Strand: Number Concepts: Addition Student Organization: Individual Skill: Computing

Try to get from 'start' to 'end' in the maze below by joining lines to connect numbers beside each other. The numbers you join, however, should add to 10 – the number in the circle.


Problem 45 – Circling Numbers

Grade: 3 or above Strand: Number Concepts: Addition Student Organization: Individual Skill: Computing



The numbers above are to be inserted into the empty circles in the diagram above. When any three adjoining numbers are added, you should not get **3**, **6** or **9**.

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Problem 46 – Stepping Stones

Grade: 6 Strand: Number Concepts: Multiples, factors Student Organization: Small groups of 5 – 6 Skills: Reasoning, justifying, predicting

A ring of stepping stones has 14 stones in it, as shown in the diagram below.



A girl hops around the ring, stopping to change legs every time she has made three hops.

- 1. How many times does she need to hop around the ring before she has stopped to change legs on each of the 14 stones?
- 2. The girl now hops around the ring, stopping to change feet every time she has made four hops. Will she eventually stop on each of the 14 stones if she continues long enough? Justify your answer.
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Problem 47 – Triangle Problems

Grade: 4 or above Strands: Number Student Organization: Pairs Skills: Computing, hypothesizing

In each triangle below, the numbers on the outside are used to determine the number inside – the same rule is used in each triangle.

1. What number is to be placed in the empty triangle?



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2. Use the rule you have observed so far to complete the following triangles.



Problem 48 – Factors

Grade: 3 or above Strand: Number Concepts: Factors, multiple Student Organization: Small groups of 5 – 6 Skills: Computing, predicting

The number 12 has six factors: 1, 2, 3, 4, 6 and 12. Four of these factors are even (2, 4 6 and 12) and two are odd (1 and 3).

- 1. Find some factors that have all of their factors, except 1, even.
- 2. What sequence of numbers has this property?

Extension

- 1. Find some numbers that have exactly half of their factors even.
- 2. Describe the sequence of numbers that has this property.



Problem 49 – Discs

Grade: 3 or above Strands: Number Concepts: Addition Student Organization: Pairs Skills: Computing, reasoning

Here are two circular cardboard discs – Disc A and Disc B.



A number is written on each disc - 7 on Disc A and 10 on Disc B. There is another number written on the reversed side of each disc. By tossing the two discs in the air and then **adding** together the two numbers which land face up, any of the following numbers can be produced:

11 12 16 17

1. Work out what numbers are on the reverse side of each disc.

2. Try to find a different solution to the problem.

Extension

1. Suppose, the numbers on one side of the discs are as shown below:



The following **products** can be produced if the numbers are multiplied after the discs are tossed in the air:

3 12 5 20

What numbers are on the other sides of the discs?

Problem 50 – Target

Grade: 4 or above Strand: Number Concepts: Multiplication, subtraction Student Organization: Pairs Skill: Computing

On a calculator, you are able to use the following keys only:



You can press them as often as you like.

- 1. Can you find a sequence of key presses to display each of the numbers from 0 10 on the calculator screen?
- 2. How many ways can you find for each number?

Problem 51 – Flower Beds

Grade: 5 – 6 **Strands**: Geometry, number, algebra **Concept**: Hexagon **Student Organization**: Small groups of 5 – 6 **Skills**: Predicting, justifying, generalizing

Hexagonal tiles are used to surround flower beds in a garden. In the design below, four flower beds (shown in red) are surrounded by 18 tiles. Other flower beds are to be added and they, too, are to be surrounded by hexagonal tiles.



- 1. Determine the number of tiles needed if there were:
 - a) 5 flower beds
 - b) 6 flower beds
 - c) 9 flower beds
- 2. Use any pattern you see to predict the number of tiles needed for 25 flower beds
- 3. How many tiles would be needed for n number of flowerbeds?

Problem 52 – Pond Borders

Grade: 5 or 6 **Strand**: Geometry and measurement **Concepts**: Area, perimeter, square **Student Organization**: Small groups of 5 – 6 **Skills**: Predicting, generalizing

Quinton works in a Home Centre that sells square pools and stone slabs to surround them. Each slab is in the shape of a square with each side being 1m in length. The customers tell Quinton what size pools they want and he works out the number of slabs to surround the pool. In the picture below, for example, the length of each side of the pool is 2 metres and 12 slabs are needed to surround it.



- 1. Determine the number of slabs needed to surround a pool where the length of each side is
 - a) 3 metres
 - b) 4 metres.
- 2. Complete a table for slabs needed for any square pool up to 12 metres in length.
- 3. Predict the number of slabs needed for a pool with each side measuring 100 metres.

Extension

1. Predict the number of slabs needed for a pool with sides of $15 \text{ m} \times 16 \text{ m}$.

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Problem 53 – Canvas

Grade: 1 Strand: Number, geometry Concepts: Quarter, half, circle, triangle Student Organization: Pairs Skills: Estimating, describing, spatial reasoning

Last night you drew the picture above in your book. Now you have to describe it to your friend over the phone what the drawing looks like so he can draw it as well. He cannot see your drawing so you have to be as precise as possible.



Problem 54 – Teacher's Banner

Grade: 1 Strand: Geometry Student Organization: Individual Skill: Justifying

You and your classmates are making a banner for your teacher. The banner you have is shown below.



You have been asked to decorate it with the following items.



- 1. Create a pattern on the banner using these items.
- 2. Explain the pattern that you have created.

Problem 55 – The Coin Problem

Grade: 3 Strand: Number Concept: Addition Student Organization: Small groups of 3 – 4 Skill: Computing

Mary has \$32 in coins as follows:



She wants to swap them out for other coins.

- 1. Which coins does she get, if she swaps them for:
 - a) 7 coins?
 - b) 8 coins?
 - c) 10 coins?
 - d) 3 different types of coins?
- 2. What is the least number of coins that Mary can use to make up her \$32?



Problem 56 – Cross Out 8

Grades: 4 – 6 **Strand**: Number **Student Organization**: Pairs **Skill**: Strategizing

This is an activity for two persons.

On a sheet of paper, a member from the pair writes the numbers 1 – 8 across a piece of paper as show below.

1 2 3 4 5 6 7 8

Each person in the pair will take turns to cross out one or two numbers on the list. The aim of the activity is to be the member of the pair who crosses out the last number left in the list. There are two rules to follow.

- a) When it is your turn to play you can cross out one or two numbers from the list you do not have to continue crossing the amount that you started with on each turn that you get.
- b) If you choose to cross two numbers, they must be beside each other no other number must separate them, not even if that number is already crossed.

Decide who starts first and try to find a winning strategy.

- 1. Suppose there were 9 numbers in the list instead of 8, how would your winning strategy change?
- 2. Describe the winning strategy for any amount of numbers in the list.

Extensions

- 1. Suppose, the person who crosses out the **last** number **loses** how would the winning strategy change?
- 2. Investigate what happens if all the rules remain the same, but the numbers are in a circle (as shown below); in this case, 8 and 1 are now 'beside' each other.



Problem 57 – Circle Nim

Grades: 4 – 6 **Strand**: Number **Student Organization**: Pairs **Skill**: Strategizing

Place nine counters in a circle, as shown below. Play against someone and remove one or two counters when it is your turn to play. The only rule is that when it is your turn to play, if you take two counters, then you must ensure that they are next to each other – no counter or open space should separate them. The winner is the one who takes up the last counter. Find a winning strategy.



Extensions

- 1. Suppose the person who removes the last counter loses, does the winning strategy change?
- 2. What if we had 10 instead of 9 counters, would the winning strategy change?
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Problem 58 – Patty Please

Grade: 4 or above **Strand**: Number **Concepts**: Addition, subtraction **Student Organization**: Small groups of 5 – 6 **Skill**: Justifying, computing

At the school's Tuck Shop, patty prices are as follows:

Cheese Patty	\$110
Beef Patty	\$90
Soy Patty	\$110
Full –House	\$180
Veg. Patty	\$120

- 1. Tammy spent \$1,700 buying patties. She bought more than one of each. How many of each could she buy? How many in all did she buy?
- 2. Karen bought three patties. She spent \$320, which three did she buy? Is there another combination possible?
- 3. Michael has a particular amount of money. He bought three types of patties and received \$30 change. If instead he had bought 2 other types of patty, he would have received \$60 change from the money.
 - a) How much money does he have?
 - b) Verify this number.



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Problem 59 – The Area Problem

Grade: 4 or above Strand: Measurement Concepts: Area, perimeter Student Organization: Individual Skills: Computing, justifying

Pauline has 16 metres of wire fencing to make a rectangular garden.

- 1. What are the possible dimensions of her garden?
- 2. What is the largest area that Pauline's garden may have?
- 3. What is the smallest area that Pauline's garden may have?



Problem 60 – Text-a-holic

Grade: 3 or above Strand: Number Concept: Addition Student Organization: Individual Skill: Computing

Michelle loves to send text messages. If she wants to send text messages to cellular phones on the LIME network, she has to pay \$5; however, if she sends messages to cellular phones on the Digicel network, she pays only \$3. In one night she spends \$42 on text messages.

- 1. How many messages did she send to each type of network?
- 2. Are there any other possible answers?



Problem 61 – The Bookshelf Problem Grade: 6 Student Organization: Small groups of about 5 – 6 Skills: Predicting, modelling

A number of books are to be placed on shelves. If there are three books – Red, Yellow and Blue – then there are six possible arrangements as shown below:



- 1. How many different arrangements are there if four books are to be placed on the shelf?
- 2. Determine the number of arrangements possible for eight books.
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Problem Number 62 – Magic Squares

Grade: 6

Strand: Number, Geometry **Concepts**: Addition, diagonal, horizontal, column, row **Student Organization**: Pairs/Small groups of 5 – 6 **Skills**: Computing, hypothesising, generalizing

Below is a 3×3 square.



1. Insert the numbers 1 – 9 in it so that the sum of each row, column and diagonal is the same.

Such a completed square is called a magic square and the sum you obtain for each column, row and diagonal is called its magic number.

- 2. Now try to create a magic square using the numbers from
 - a) 2-10
 - b) 3–11
- 3. Develop and explain a system for completing any 3×3 magic square even if the numbers are not in counting or any sequential order.
- 4. Write out two separate lists of 9 random, unrelated numbers that could produce magic squares. Explain how you know that these numbers could produce magic squares.
- 5. Use the system developed in parts (b) and (c) above to create a magic square:
 - a) with a magic number of 27;
 - b) using only odd numbers;
 - c) using only even numbers.
- 6. Starting with any completed 3 × 3 magic square, which one of the following changes would result in a magic square?
 - a) Adding the same number to each entry already in the square.
 - b) Doubling each entry in the magic square.
 - c) Squaring each entry in the magic square.

Extensions

Try to make a 3×3 magic square using only

- 1. prime numbers; and
- 2. square numbers.



APPENDIX 1 – SELECTED SOLUTIONS

Problem 1- Number Chains

1. Using the rules outlined we can create the following number chain using 8 as the starting number.



- 2. From the chain above, we notice that except for the starting number (8), the numbers in the chain are repeated at regular intervals.
- 3. Four other chains, using the prescribed rules, are explored below using 13, 7, 6, and 10 as starting numbers.



4. From the chains above it may be observed that once there is a 1 in the chain the numbers 4, 2, and 1 are repeated at regular intervals. In order to ensure that this happens we need to ensure that we choose starting numbers that when repeatedly divided by two will give an even number until eventually 1 is obtained. This means that the number we choose to begin our chain should be a power of 2.

Problem 2- Cutting Squares

1. To determine the perimeter of the new figure, we will simply count the number of unit squares along each side.



So when we start with a **4 X 4** square, the perimeter of the new figure is:

(3 X 6) + 4 units = 22 units.

2. (a) – (b) The diagrams below show a 5 X 5 and 6 X 6 rectangle rearranged as directed.



 $(6 \times 6) + 4 \text{ units} = 40 \text{ units}.$

(d) In general, if we rearrange an $n \times n$ square, its perimeter will be: $[(n-1) \times 6] + 4$

Problem 4 – Perfect squares

- 1. When we examine the results obtained so far, we see that the last two digits are always 25.
- 2. As shown in the table below, for each of the numbers being squared, the first digit is multiplied by the number that is 1 more than it self.

Numbers	5	<mark>1</mark> 5	<mark>2</mark> 5	<mark>3</mark> 5	<mark>4</mark> 5	<mark>5</mark> 5	<mark>6</mark> 5	<mark>9</mark> 5
Result when Squared	25	225	625	1225	<mark>20</mark> 25	<mark>30</mark> 35	4225	<mark>90</mark> 25

3. We can use the observations from parts a. & b. to complete the table as shown above.

Problem 9 - All in a Circle

We can create a drawing to model a circle with 12 persons as shown below.



- 1. From the diagram we can see that:
 - a) The person across from the 4th person is the 10th person.
 - b) The person across from the 7th person is the 1st person.

Taking a closer look at the diagram there are a few things that may be observed.

c) There are 6 persons from the 4th to the 10th person. Note that that 6 is half of 12, the total number of persons in the circle.

- d) The person across from the 12th person is the 6th person. If we draw a line connecting these two, the circle is divided equally with the same number of persons on either side of the line.
- e) The 4th person is four places away from the 12th (last) person. The person who is directly across from the 4th person (the 10th person) is four places away from the 6th (middle) person.
- 2. Using the observations from the previous task we can make the following observations:
 - a) There are 25 persons from the 9th person to the 34th person. There are, therefore, 50 persons in the circle. Put another way, the 9th person should be 9 places away from the last person in the circle, so the 34th person should be 9 places away from the middle person. The middle position can be found by subtracting 9 from 34.

34 – 9= 25

By multiplying by 2, we can determine the number of persons in the circle.

25 × 2=50

So there are 50 persons in the circle.

b) To determine the person across from the 15th person, add 15 and 25 to get 40. So the 40th person is directly across from the 15th person.



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Problem 11- The 400 Problem



- 1. Since the answer to problem 1 is the same as the answer to problem 2, we can compare the values in each place value position to determine the values for *a* and *b*. Let us look at the ones placed in both problems.
 - a) In problem 1 we have 0 4 in the ones place.
 - b) We know that we will have to regroup, so we will regroup and subtract to get 6 as shown below:



- c) Since the answer to problem 2 is the same as the answer to problem 1, we know that b 0 must also give us 6. Since b 0 = b, we now know that b = 6.
- d) Similarly, when we look at the tens place in problem 1 we have 9 b, we now know that b is 6, so we have 9 6 which gives us 3. Once again, since the solution to both problems is the same, we know a 0 = 3, and so a = 3.



So a = 3 and b = 6.

2. By substituting the values of *a* and *b* in the original problems, we can verify that the values found are correct.



- 3. Using a similar approach as before we can determine that if:
 - a) The problem were a 500-problem, the values of *a* and *b* would be 4 and 5 respectively.
 - b) The problem were a 600 problem, the values of a and b would be 5 and 4 respectively.
- 4. Using the approach in part 1, we can obtain the following values for a and b:

Number of hundreds	4	5	6	7	n
а	3	4	5		
b	6	5	4		

When we look at the results recorded in the table, there are a few observations we can make.

- a) The value of *a* is steadily increasing by 1 as the number of hundreds increases.
- b) The value of b decreases by 1 as the number of hundreds increases.
- c) In each instance a + b = 9.
- d) The value of a is one less than the number of hundreds and the sum of the number of hundreds and b is always 10.

Number of hundreds	4	5	6	7	n
a	4 – 1 = 3	5 – 1 = 4	6 – 1 = 5	7 – 1 = 6	n – 1
Ь	10 – 4 = 6	10 – 5 = 5	10 – 6 = 4	10 – 7 = 3	10 – <i>n</i>

In general, for an *n* hundred problem, a = n - 1, b = 10 - n

Problem 13 – Trading Places

We can immediately substitute the value of A in the equations as we were told that A = 5. The equations now read:

	Pr	oblem	1		Problem 2						
	F	5	Ν		R	5	Ι	Ν			
+	Μ	5	N	+	Ν	0	0	Ν			
	5	I	R	Μ	0	Т	0	R			

Bearing in mind that N is less than 5, and given that in the tens column in problem 1 we have 5 + 5 = I, we can deduce that I = 0, since 5 + 5 = 10 (we would have to 'carry' the 1). And the problems now read:

	Pro	oblem	1		<u>Problem 2</u>						
	¹ F	5	Ν		R	5	0	Ν			
+	Μ	5	Ν	+	Ν	0	0	Ν			
	5	0	R	Μ	0	Т	0	R			

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If we look now at problem 2, we see that R + N = MO. Since each letter represents a single digit, we know that the greatest sum of any two letters is 9 + 9 = 18, so M can only be 1. If we substitute M = 1, into problem 1, we now have:

$$\frac{Problem 1}{^{1}F \quad 5 \quad N}$$

$$\frac{+ 1 \quad 5 \quad N}{5 \quad 0 \quad R}$$
From this we can see that F = 3.

Since N < 5, and other letters have already been assigned to 1 (M) and 3 (F), N is equal to either 2 or 4. In the thousands place of problem 2, we note that:

- a) R + N is equal to a two-digit number
- b) R = N + N = 2N.

Therefore, R + N = 2N + N = 3N.

If N = 2, then 3N = 6 (which is not a two digit number). Therefore, N = 4 and R = 8. We now have:

	Pr	oblem	1				Proble	e <u>m 2</u>	
	3	5	4			8	5	0	4
+	1	5	4		+	4	0	0	4
	5	0	8		1	0	Т	0	8

Clearly, O = 2 (since, 8 + 4 - 12) and therefore, T = 7.

	Pr	oblem	1		<u>Problem 2</u>						
	3	5	4		8	5	0	4			
+	1	5	4	+	4	2	2	4			
	5	0	8	1	2	7	2	8			

Digit	0	1	2	3	4	5	7	8
Letter	Ι	М	0	F	Ν	A	Т	R

Problem 18 – Counting Rectangles

1. Shape 4 is shown below.



So in all, **Shape 4** has a total of 10 shapes.

2. The 5th shape will have 15 rectangles. In order to make our prediction we can record the information we have so far in tabular form to look for a pattern.



From the table above we can see that the number of rectangles for the nth shape is the sum of whole numbers from 1 to n.

3. Using the same form we can predict the number of rectangles in the first 12 shapes as shown in the table below.

Shape No.	1	2	3	4	5	6	7	8	9	10	11	12
No. of rectangles	1	3	6	10	15	21	28	36	45	55	66	78

Problem 21 – T-Patterns

We have recreated the patterns below to show that each pattern has 2 dots (coloured in red) around which other dots (coloured in green) have been placed.

Pattern 1 has 5 dots – 2 red dots, 1 green dot below the red dots, 1 green dot to the left and 1 green dot to the right of the red dots.

Pattern 2 has 8 dots – 2 red dots, 2 green dots below the red dots, 2 green dots on the right and 2 green dots on the left

Pattern 3 has 11 dots – 2 red dots, 3 green dots below the red dots, 3 green dots to the right and 3 green dots to the left.



- The 65th pattern would have 197 dots 2 red dots plus 65 green dots flanking them 65 below, 65 to the right and 65 to the left
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- 2. We can note that:
 - a) The pattern number indicates the number of green dots flanking the red dots:
 - In pattern 1, only 1 dot is on each of the 3 sides of the 2 red dots
 - In pattern 2, there are 2 dots on each of the 3 sides of the 2 red dots
 - In pattern 3, there are 3 dots on each of the 3 sides of the 2 red dots
 - In pattern 65, there are 65 green dots on each of the 3 sides of the 2 red dots
 - In pattern *n*, therefore, there will be *n* dots on each of the 3 sides of the 2 red dots Therefore, total number of dots = 3n + 2
 - b) We can also note that:
 - the number of dots horizontally is always 1 more than twice the pattern number. That is 2n + 1
 - the number of dots arranged vertically is 2 more than the pattern number. That is n + 2218 = 3n + 2218 = 3n + 2. Therefore 3n = 2163n = 216; n = 72n = 72.

Problem 22 – Exploring Polygons

1. In order to answer the questions asked we can draw a polygon with six sides



From the diagram we can see that

- a) Three Lines can be drawn through the polygon if it has six sides.
- b) The three lines divide the polygon into four triangles; ABC, ACD, ADE, and AEF

Before we can make a generalization we can try a few more polygons and record our findings in tabular form.



Number of sides			$\left \right\rangle$	()	$\left \right\rangle$		
Number of l ines	4-31	5-32	6-33	7-34	8-35	9-36	n - 3
Number of triangles	2 4-	2 3 5 - 2	2 4 6-2	2 5 7 - 2	8 – 2 6	9-2 7	n – 2

For a polygon which has 56 triangles:

- a) n-2 = 56; n = 55.
- b) Number of lines = n 3 = 58 3 = 55.

Problem 23 – Last One Standing

1. We can verify that when 5 persons are in the room, the last one standing in the circle is the 3rd person by modelling the problem. We may do this with actual persons, objects or using a diagram as shown below.



As we go around the circle we may simply cross out those persons who are out. So that at the end of the first round we have:



And at the end of the second round, we have:



So when there are five persons in the room the last one standing in the circle is indeed the person.

2. Choosing and modeling simpler numbers produces the table shown below:

No. of persons in the circle	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Last person standing	1 st	1 st	3 _{rd}	1 st	3 _{rq}	5 th	7 th	1 st	3 _{rd}	5 th	7 th	9 th	11 th	13 th	15 th	1 st	3rq

If we examine the pattern created by the numbers indicating what position the last person standing is in, we observe that:

- a) The numbers are all odd, listed in counting sequence starting with 1;
- b) The pattern repeats and then expands the colour in the table changes each time the number pattern restarts at 1;
- c) The pattern restarts at 1 whenever the number of persons in the circle is a power of 2 (1, 2, 4, 8, 16, ...); and
- d) Right before the pattern restarts the winner is the last person in the circle.
 Since 64 is a power of 2, when there are 64 persons in the circle the winner will be the first person.

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e) In order to predict when the 31st person will when we need to find out how close it is to a power of 2. The powers of two are: 1, 2, 4, 8, 16, 32, 64.

Clearly, 32 is the nearest power of 2 to 31. We know that pattern will restart when there are 32 persons in the circle and since 31 directly precedes 32, the 31st person will win when there are 31 persons in the circle.

Problem 27 – The Weighing Problem

At first glance it would appear that if we had four weights we could only accurately measure four masses. Or, at most, we might see that we could measure other masses by adding weights. So, if we had a 1g weight and a 3g weight we could place both weights on one side of the scale and a mass of 4g (1g +3g) would need to be placed on the other side of the scale in order for the scale to balance.



In this way we could combine any two, three or even four of the weights we had. It means, then, that we did not need, say, a 40g weight, we could have, instead, chosen four weights that would total 40g, for example 10g, 15g, 8g and 7g.



We could also measure any mass which was the difference of the masses of any two weights or two combinations of weights. Keep in mind that in order for the scales to be balanced the weight on either side of the scale must be the same. If we had unequal masses on either side of the scale, the scale would only balance when the difference was placed on the side with the smaller mass.

For example, if a 1g weight is placed on one side and a 3g weight on the other side then the scale would only balance when a mass of 2g is placed on the side with the 1g weight, if it does not balance then the mass is not 2g.



This also opens up the options for the weights we may choose. So, for example, instead of choosing a 3g weight we may choose, instead, two weights whose masses have a difference of 3g for example we may choose 12g and 15g. We may also find that we can use masses that when combined, have differences of 3g. For example, if we used weights with masses of 2g, 10g, and 15g, we could place both the 2g and the 10g weights on one side of the scale which would give us a total of 12g on that side and place the 15g on the other. A mass of 3g would need to be placed on the side with 12g in order for the scale to balance.



In order to maximize the number of masses we are able to weigh, we would need to select numbers that when combined either by adding or subtracting would not duplicate any masses. For example, if we used a 10g and an 11g mass, we would not then also use a 1g mass as this would duplicate the difference and we would lose the chance of having another potential weight.

Using these rules if we use 1g, 7g, 11g, and 21g weights, we are able to accurately measure 34 of the 40 masses.

1	2 (21-1-11-7)	3 (11-1 -7)	4 (11-7)	5 (11-7+1)	6 (7-1)	7	8 (7+1)	9 (21-11-1)	10 (21-11)
11	12 (11+1)	13 (21-7-1)	14 (21-7)	15 (21-7+1)	16 (21-11+7-1)	17 (21- 11+7)	18 (21-11+7+1)	19 (11+7+1)	20 (21-1)
21	22 (21 +1)		24 (21+11-7-1)	25 (21+11-7)	26 (21+11-7+1)	27 (21+7-1)	28 (21+7)	29 (21+7+1)	
31 (21+11-1)	32 (21+11)	33 (21+11+1)					38 (21+11+7-1)	39 (7+11+21)	40 (1+7+ 11+21)

Problem 28 – The Triangular Garden Plot

Note that the number of corns that Jackson plants is really determined by the sequence of odd numbers – the first odd number is 1 and he plants 1 corn in the first row; the second odd number is 3 and he plants 3 corns in the 2nd row and the 3rd odd number is 5 and he plants 5 corns in the 3rd row.

- 1. The number of corns that he plants in the 12^{th} row will simply be the 12^{th} odd number 23. This is determined using the formula 2n - 1, where n represents the position of the odd number.
- 2. The sum of the corn seed in the first 12 rows is the sum of the first 12 odd numbers. Investigating odd numbers reveal that odd numbers and squaring have a relationship.
 - a. The sum of the first **2** odd numbers = 1 + 3 = 4, but $2^2 = 4$.
 - b. The sum of the first **3** odd numbers = 1 + 3 + 5 = 9, but $3^2 = 9$.
 - c. The sum of the first 4 odd numbers = 1 + 3 + 5 + 7 = 16, but $4^2 = 16$.
 - d. Therefore, the sum of the first 12 odd numbers = $12^2 = 144$.
- 3. The *n*th row will have 2n 1 corn seeds.
- 4. After the *n*th row, he would have planted n^2 corn seeds.
Problem 30 – Chess Board Squares

1. A 3×3 board is shown below – letters have been put inside each small square for ease of reference.

А	В	С
D	Е	F
G	Н	I

It has

- c) nine 1 × 1 squares A, B, C, D, E, ..., I;
- d) four 2 × 2 squares ABED, BCFE, DEHG, EFIH;
- e) one 3×3 square made up of all the letters; and
- f) a total of 14 squares (9 + 4 + 1 = 14).
- 2. A 4×4 square is shown below.

А	В	С	D
E	F	G	
I.	J	K	L
	Ν	0	Р

The 4×4 square is made up of

- a) sixteen 1×1 squares A, B, C, D, E, ..., P;
- b) nine 2 × 2 squares ABFE, BCGF, CDHG, EFJI, FGKJ, GHLK, IJMN, JKON, KLPO;
- c) four 3 \times 3 squares ABCGKJIE, BCDHLKJFB, EFGKONMI, FGHLPONJ; and
- d) one 4×4 square made up of all the letters.

In total, it has 30 squares (16 + 9 + 4 + 1).

3. Given that an 8×8 square is so large, let us see if we can find a pattern from those that we have already done. The table below may help:

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Dimension	1×1	2 × 2	3 × 3		
Number of	1	4 + 1 = 5	9 + 4 + 1 = 14		
squares	1 ²	$2^2 + 1^2$	$3^2 + 2^2 + 1^2$		

By looking at the table, it becomes clear that the number of squares is found by adding consecutive squared numbers. In each case, the largest squared number is equal to the length of a side of the square. The number of squares in an 8×8 square will be:

 $8^2 + 7^2 + 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2 = 204$

Problem 31 – Addend Pairs

- 1. The number of pairs for:
 - a) there are four pairs of numbers that add to 7. These are shown below.(0 and 7), (1, and 6), (2 and 5), (3 and 4)
 - b) 8 there are 5 pairs of numbers that add to 8 (shown below) (0 and 8), (1 and 7), (2 and 6), (3 and 5), (4 and 4)
- 2. The table below shows how many pairs of addends add to various sums.

Sums	1	2	3	4	5	6	7	8
No. of addends	1	2	2	3	3	4	4	5

Even numbers (those highlighted in green) have a different relationship from odd numbers (those highlighted in blue).

- a) If x is an **<u>even</u>** number , No. of addends = (x + 2) + 1.
- b) If x is an **<u>odd</u>** number, no. of addends = (x + 2) + 2.
- 3. a) The number 15 has 8 pairs of addends.

No. of addends = $(15 + 1) \div 2 = 8$

- b) The number 1,998 has 1,000 pairs of addends.
 No. of addends = (1,998 ÷ 2) + 1 − 1,000
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Problem 32 – Newspaper Math

- 1. Since there are 8 pages in front of page 9, then there are 8 pages behind page 34. There are therefore 32 pages in the newspaper (that is 34 + 8 = 32)
- 2. Pages 16 and 17 are in the middle of the newspaper.
- 3. Similarly, there are 10 pages in front of page 11 and hence 10 pages behind page 34. The newspaper has 44 pages in total.
- 4. A newspaper with page 9 across from page 34, as shown below, will have 9 1 = 8 pages behind page 34. In total it will have 34 + 9 1 = 42 pages.



- a) If the newspaper had page 7 across from page 34, then it would have 6 (that is 7 1) pages behind page 34. In total it would have 34 + 7 1 = 30 pages.
- b) If a newspaper had page x across from page y, then it will have x 1x 1 pages behind page y. In total, the newspaper would have y + x 1y + x 1 pages.
- 5. Total number of pages = y + x 1

50 = 5 - 1 + y46 = y

Problem 33 – Chicken Combos

Each piece of chicken – Leg (L), Breast (B), Wing (W), Rib (R) or Thigh (T) – can be had with any
of the four other pieces. The various combinations are as shown below (in the first row, we list
combinations with Leg; combinations with Rib in the 2nd row; combinations with Breast in the
third row and combinations with Wing in the 4th row):

LR	LB	LW	LT
RB	RW	RT	
BW	BT		
WR			

Note that the number of combinations decreases from one row to the next – this is to prevent double counting.

There are, therefore, 10 two-piece combinations possible:

a) with 5 pieces of chicken, note that

$$4 + 3 + 2 + 1 = 10$$
.

b) With 6 pieces of chicken, there will be 15 combinations possible:

$$5 + 4 + 3 + 2 + 1 = 15$$

Note that when there are 5 pieces of chicken, number of combinations is the sum of numbers up to 1 less than 5 (that is 4 + 3 + 2 + 1). Additionally, when there are 6 pieces of chicken, number of combinations is the sum of numbers up to 1 less than 6 (that is 5 + 4 + 3 + 2 + 1). If there are *n* number of different parts of chicken, we take the sum of numbers up to 1 less than *n*.

Problem 35 – Match Drawn

1. There are 9 possible scores for a match that ended 2 - 2:

 $0-0 \quad 0-1 \quad 0-2 \quad 1-0 \quad 1-1 \quad 1-2 \quad 2-0 \quad 2-1 \quad 2-2$

- 2. Note that for the scores above, there are 3 scores starting with 0, 3 starting with 1 and 3 starting with 2. This is because, 0 goals scored by the home team is matched with either 0, 1, or 2 goals scored by the visiting team. Similarly, 1 goal scored by the home team is matched with either 0, 1, or 2 goals scored by the visiting team and so on. If the final score is 3 3, then possibly there are:
 - a) 4 scores starting with 0(0 0, 0 1, 0 2 and 0 3)
 - b) 4 scores starting with 1
 - c) 4 scores starting with 2
 - d) 4 scores starting with 3
 - e) **16 possible half time scores**.
- 3. Consider the table below

Final score	0-0	1 – 1	2 - 2	3 – 3	4 – 4	5 – 5	6 – 6	7 – 7	n - n
Number of possible half time scores	$1^2 = 1$	$2^2 = 4$	$3^2 = 9$	$4^2 = 16$	$5^2 = 25$	$6^2 = 36$	$7^2 = 49$	$8^2 = 64$	$(n + 1)^2$

Note that

- The number of possible half time scores is always a squared number
- If the final score is 0 0, the number of possible half time score is the square of (0 + 1) that is (0 + 1)². Similarly, if the final score is 2 2, the number of possible half time score is (2 + 1)² and so on.
- If the final score is n nn n, the number of possible half time score is $(n + 1)(n + 1)^2$.
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Problem 38 – The Trapezium Problem

1. For PQRS,

Top length (PQ) = 2 units Bottom length (SR) = 4 units Slant length (QR = PS) = 2 units. Number of unit triangles = 12

For TUVW,

Top length (TU) = 1 unit Bottom length (VW) = 4 units Slant length (UV = TW) = 3 units. Number of unit triangles = 15

For EFGH

Top length (EF) = 3 units Bottom length (GH) = 5 units Slant length (EH = FH) = 2 units. Number of unit triangles = 16

- 2. To determine the number of unit triangles in any trapezium: *add the top and bottom length and then multiply by the slant length*.
- 3. (a) Number of unit triangles = 1(4 + 5) = 9
 - (b) Number of unit triangles = 2(3 + 1) = 8

Problem 51 – Flower Beds

1. Note that the first flower bed needs six tiles. However, the 2nd flower bed needs only four more flower tiles. In fact each additional flower bed needs only four tiles. This is shown below.



- a) 1 flower bed has $6 + (0 \times 4)$ tiles = 6 tiles.
- b) 2 flower beds have $6 + (1 \times 4)$ tiles = 10 tiles.
- c) 3 flower beds have $6 + (2 \times 4)$ tiles = 14 tiles.
- d) 4 flower beds have $6 + (3 \times 4)$ tiles = 18 tiles.
- e) 5 flower beds have $6 + (4 \times 4)$ tiles = 22 tiles.
- f) 6 flower beds have $6 + (5 \times 4)$ tiles = 26 tiles.
- g) 9 flower beds have $6 + (8 \times 4)$ tiles = 38 tiles.
- 2. The number of tiles needed for 25 tiles would be the 6 tiles need along with the 4 needed for each of the other 24. That is $6 + (24 \times 4) = 102$ tiles.
 - a) For *n* flower beds, we would need the following number of flower beds.

 $6 + (n - 1) \times 4 = 6 + 4n - 4 = 4n + 2$

Problem 52 – Pond Borders

- 1. The number of slabs needed for a pool
 - a) 3 meters is found as follows:
 - Each side of the pool will need 3 slabs therefore, $4 \times 3 = 12$
 - Each of the 4 corners will need a slab
 - The total number of slabs = $(4 \times 3) + 4 = 16$
 - b) 4 meters »» $(4 \times 4) + 4 = 20$
- 2. The number of slabs needed for each side of the pool will be equivalent to its dimensions. To this are added the 4 slabs needed for the corners. The table below gives the number of slabs needed for pools with dimensions of up to 12 units.

Dimensions (m)	1	2	3	4	5	6	7	8	9	10	11	12
Number of slabs	8	12	16	20	24	28	32	36	40	44	48	52

Note that:

- the number of slabs is always a multiple of 4;
- if we add 1 to the dimension and multiply this by 4, we get the number of slabs; and
- if the pool has length of 100 m, then the number of slabs needed = $101 \times 4 = 404$.

Problem 56 – Cross Out 8

The winning strategy involves creating symmetry – ensuring that your play divides the board into two congruent pieces. Essentially, this involves the following steps.

1. Playing first and crossing out 4 and 5:



- 2. On each subsequent play, ensure that you copy your opponent's play on the opposite side of the board. So, for example, if your opponent:
 - a) Crosses out 2, make sure you cross out 7; and
 - b) Crosses out 7 and 8, make sure you cross out 1 and 2.
- 3. If there are nine numbers in the list, then play first and cross out 5 since there would now be four numbers on either side. In general:
 - a) For a game with even amount of numbers, play first and cross out the two numbers in the middle that create symmetry; and
 - b) For a game with odd number of numbers, play first and cross out the one number in the middle that create symmetry.

APPENDIX 2 – GLOSSARY

Algorithms

A set of rules or step-by-step procedures to guide the completion of mathematical tasks.

Classifying

To arrange or organise according to specific characteristics or attributes, such as colour or shape.

Conjectures

Opinions formed purely on speculation or incomplete information.

Cognitive root

An easily understood concept, idea, principle or approach on which future understanding of more complex concepts can be built.

Difference

The size of the gap between two numbers and is found by subtracting the smaller from the larger.

Effective strategy

A strategy that can be employed to yield the correct answer, conclusion or desired result.

Efficient strategy

A strategy that yields a correct answer, conclusion or desired result with minimum wasted effort or expense.

Extensions

An addition or modification to a problem that the solver is asked to resolve by applying what they have already learnt from solving the original problem.

Generalizing

The process of using specific instances to make open ended statements about all other similar instances.

Guess and test

A problem-solving strategy that involves selecting likely solutions in accordance with the conditions in a problem and checking to see if these solutions are correct.

Heuristics

Strategies employed in problem-solving that facilitate the discovery of a solution without the use of pre-established algorithms.

Horizontal difference

The difference between values in adjoining or specified columns in a table.

Hypothesising

The process of forming a logical explanation for an observation based on limited evidence to be later tested.

Impulsivity

The desire to act or jump to conclusions prematurely.

Invented strategies

Strategies devised by students outside of the traditional algorithms to solve problems or complete tasks.

Justifying

Presenting reasonable arguments to support conclusions.

Logical reasoning

A course of argument that develops soundly on the basis of available evidence, rational conclusions and suppositions.

Models

Representation of objects or systems in the real world that allows for easy investigation to understand how things operate in the real world.

Modelling

Method of using mathematics (for example equations, tables and graphs) to describe real life situations.

Non-routine problems

A task or activity for which students have no prescribed method of obtaining a solution, nor do they believe that there is an accessible method that could be employed in obtaining a solution.

Open problems

An ambiguous problem in which either the solution obtained or the approach taken to solve the problem can differ from one student to the next.

Predicting

The process of stating beforehand the outcome of future events by analysing the outcomes of previously observed events.

Problem

A task having no immediate solution which requires the problem-solver to think and which the problem-solver accepts the challenge of solving.

Problem solving

The process of employing invented or contrived strategies to solve a problem.

Proof

A logical sequence of statements (steps) to establish the truth of a mathematical result.

Readiness

The extent to which students have acquired the necessary skills and attitude to complete a task.

Reliability

The extent to which an instrument consistently measure s/performs its intended or required function

Representation

A model or image in the form of tables, graphs or equations depicting something in the real world. In general, representation refers to how mathematical thinking is manifested in symbols, statements, graphs, charts, drawings or other models.

Rubric

A guideline for rating students' performance.

Routine problems

Tasks which test students' ability to use standard algorithms. They are usually in the form of traditional "worded problems".

Scaffolding

A teaching strategy in which students are supported by teachers as they are being introduced to complex ideas. The support is gradually reduced as students gain competence until they are able to work independently.

Subjectivity

Judgment based on or influenced by personal opinions, feelings, and biases rather than solely on the facts.

Transfer

A students' ability to use skills or knowledge gained in one area to seek solutions in another area.

Validity

The extent to which a test or any instrument accurately measures the intended objectives that should have been achieved.

Vertical difference

The difference between values in adjoining or specified rows in a table.

Volitional strategies

Students' ability to act independently of the teacher or other sources of guidance (such as the textbooks or previously worked examples of similar problems) and create solution paths when faced with novel problems.

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